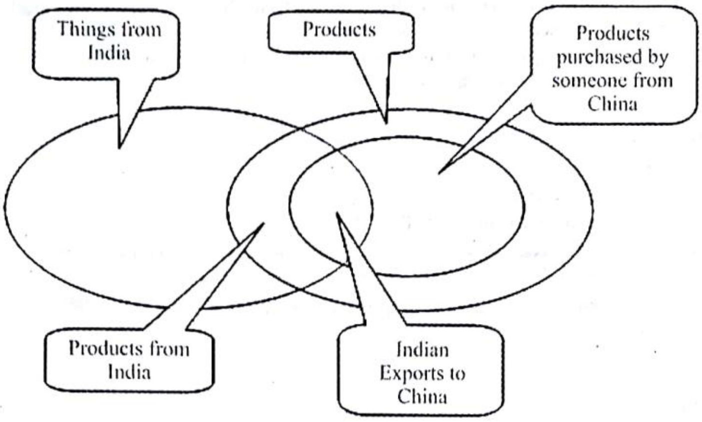
PART B



11

CHAPTER

Sets, Relations and Functions

INTRODUCTION

The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

In this chapter, we discuss some basic definitions, concepts and operations involving sets.

In mathematics, we come across many relations such as number m is less than the number n, line l is parallel to m. In this chapter, we will learn how to link objects of one set with the object of other set. Finally we will learn about special type of relations called functions. Concept of functions are very important in mathematics.

SETS

A set is a well-defined collection of different objects.

In everyday life, we often speak about the collection of objects of particular kind such as a cricket team, the river of India, the vowels in the English alphabet etc. Each of these collection is well-defined collection of objects in the sense that we can definitely decide whether a given particular object belongs to a given collection or not. For example, we say that 10 does not belongs to the given collection of all odd natural numbers. On the other than, 15 belongs to this given collection.

The following points may be noted :

(i) Objects, elements and members of a set are synonymous terms.

(ii) Sets are usually denoted by capital letters A, B, C, D, E, F, etc.

(iii) The elements of a set are represented by small letters a, b, c, d, e, f, etc.

(iv) Each element in a set comes only once i.e. repetation of any element is not allowed.

If a is an element of a set A, we say that " a belongs to A". The Greek symbol $\in$ (epsilon) is used to denote the phrase 'belongs to'. Thus, we write $a \in A$. If 'b' is not an element of a set A, we write $b \notin A$ and read "b does not belong to A".

Thus, in the set V of vowels in the English alphabet, $a \in V$ but $b \notin V$. In the set P of prime factors of 30, $3 \in P$ but $15 \notin P$.

REPRESENTATIONS OF SETS

There are two methods of representing a set :

(i) Roster or tabular form (ii) Set-builder form.

ROSTER OR TABULAR FORM

(i) In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all even positive integers less than 7 is described in roster form as {2, 4, 6}.

(ii) In roster form, the order in which the elements are listed is immaterial.

SET-BUILDER FORM

The set $\{a, e, i, o, u\}$ in roster form can be written as set in builder form as {: x is a vowel in English alphabet}. Here the set written in set builder form read as 'x' is an element of the set such that x is a vowel of English alphabet'. Here the colon (:) read as 'such that'. And in set-builder form after ':' a common property which posses all the elements of the set is written.

COMPARISON TABLE

Statement	Roster form	Set builder form
(1) The set of currencies used in USA, England, Japan, Germany and Russia.	{Dollar, Pound, Yen, Euro, Rouble}	$\{x : x$ is the currencies used in USA, England, Japan, Germany and Russia$\}$
(2) The set of Capital of Kerala, Karnatka, Tamilnadu, Andhra Pradesh and Gujarat.	{Thiruvananthapuram, Banglore, Chennai, Hyderabad and Gandhi Nagar}	$\{x : x$ is the capitals of Kerala, Karnatka, Tamilnadu, Andhra Pradesh and Gujarat$\}$
(3) The set of all distinct letters used in the word student.	$\{s, t, u, d, e, n\}$	$\{x : x$ is the distinct letters used in the word student.$\}$
(4) The set of all the states of India beginning with the letter A.	{Andhra Pradesh, Arunachal Pradesh, Assam}	$\{x : x$ is the state of India beginning with the letter $A\}$
(5) The set of six presidents of India since 1980.	{Neelam Sanjeeva Reddy, Gyani Zail Singh, Radha Swami Venkat Raman, Dr. Shankar Dayal Sharma, K.R. Narayan, A.P.J. Abdul Kalam}	$\{x : x$ is the presidents of India since 1980$\}$
(6) The set of all natural numbers between 11 and 15.	{12, 13, 14}	$\{x : x \in N, 11 < x < 15\}$

STANDARD SYMBOLS OF SOME SPECIAL SETS

N : Set of all natural numbers
Z : Set of all integers
Q : Set of all rational numbers
R : Set of all real numbers
Z^+ : Set of all positive integers

Q^+ : Set of all positive rational numbers, and
R^+ : Set of all positive real numbers.

The symbols for the special sets given above will be referred to throughout the chapter.

TYPES OF SETS

EMPTY SET

A set which does not contain any element is called the empty set, null set or void set.
The empty set is denoted by the symbol ϕ or { }.
Given below are few examples of empty sets.

(i) If $A = \{x : 1 < x < 2, x \text{ is a natural number}\}$. Then A is the empty set, because there is no natural number between 1 and 2.
(ii) If $B = \{x : x^2 - 2 = 0 \text{ and } x \text{ is rational number}\}$. Then B is the empty set, because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.
(iii) If $C = \{x : x \text{ is an even prime number greater than } 2\}$. Then C is the empty set, because 2 is the only even prime number.
(iv) If $D = \{ x : x^2 = 4, x \text{ is odd} \}$. Then D is the empty set, because the equation $x^2 = 4$ is not satisfied by any odd value of x.

FINITE AND INFINITE SETS

A set which is empty or consists of a definite number of elements is called finite otherwise, the set is called infinite.
Consider some examples :

(i) Let W be the set of the days of the week. Then W is finite.
(ii) Let S be the set of solutions of the equation $x^2 - 16 = 0$. Then S is finite.
(iii) Let G be the set of points on a line. Then G is infinite.

When we represent a set in the roster form, we write all the elements of the set within braces { }. It is not possible to write all the elements of an infinite set within braces { } because the numbers of elements of such a set is not finite. So, we represent the infinite set in the roster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots. For example, $\{1, 2, 3, \ldots\ldots\ldots\}$ is the set of natural numbers, $\{1, 3, 5, 7, \ldots\ldots\}$ is the set of odd natural numbers, $\{\ldots\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\}$ is the set of integers. All these sets are infinite.
All infinite sets can not be described in the roster form. For example, set of all real numbers can not be described in roster form.

EQUAL SETS

Two sets A and B are said to be equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be unequal and we write $A \neq B$.

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 1, 4, 2\}$. Then $A = B$.
(ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, since 2, 3 and 5 are the only prime factors of 30 and also these are less than 6.

SINGLETON SET

A set, consisting of a single element is called a singleton set. The sets $\{0\}$, $\{5\}$, $\{-7\}$ are singleton sets.
$\{x : x + 6 = 0, x \in Z\}$ is a singleton set, because this set contains only integer namely, -6.

SUBSETS

A set A is said to be a subset of a set B if every element of A is also an element of B. Here set B is called superset of set A. In other words, $A \subset B$ if whenever $a \in A$, then $a \in B$. It is often convenient to use the symbol "$\Rightarrow$" which means implies. Using this symbol, we can write the definition of subset as follows: $A \subset B$ if $a \in A \Rightarrow a \in B$
We read the above statement as "A is a subset of B if a is an element of A implies that a is also an element of B". If A is not a subset of B, we write $A \not\subset B$. For example :

(i) The set Q of rational numbers is a subset of the set R of real numbers, so we write $Q \subset R$.
(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A so we write $B \subset A$.
(iii) Let $A = \{1, 3, 5\}$ and $B = \{x : x \text{ is an odd natural number less than } 6\}$. Then $A \subset B$ and $B \subset A$ and hence $A = B$.
(iv) Let $A = \{ a, e, i, o, u\}$ and $B = \{ a, b, c, d\}$. Then A is not a subset of B, also B is not a subset of A.

Important Points About Subsets

(i) Every set is a subset of itself.
(ii) The empty set is a subset of every set.
(iii) The total number of subsets of a finite set containing n element is 2^n.

INTERVALS AS SUBSETS OF SET OF REAL NUMBERS (R)

Let $a, b \in R$ and $a < b$. Then the set of real numbers $\{ y : a < y < b\}$ is called an open interval and is denoted by (a, b). All the points between a and b belong to the open interval (a, b) but a, b themselves do not belong to this interval.

The interval which contains the end points also is called closed interval and is denoted by $[a, b]$. Thus

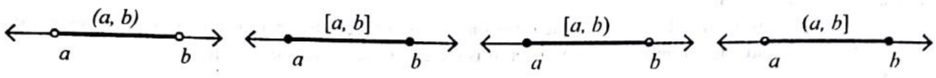
$$[a, b] = \{x : a \le x \le b\}$$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \le x < b\}$ is a semi open interval from a to b, including a but excluding b.

$(a, b] = \{x : a < x \le b\}$ is a semi open interval from a to b including b but excluding a.

These sets can be shown by the dark portion of the number line.



POWER SET OF A SET

The collection of all subsets of a set A is called the power set of A. It is denoted by $P(A)$. In $P(A)$, every element is a set.

If A has n elements then its power set has 2^n elements.

Let $A = \{a, b\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

UNIVERSAL SET

If there are some sets under consideration, and out of these sets, there is a set which is the superset of each one of the given sets. That is all other sets under consideration are subsets of this set. Such a set is known as the universal set, denoted by U.

For example, (i) In the context of human population studies, the universal set consists of all the people in the world.

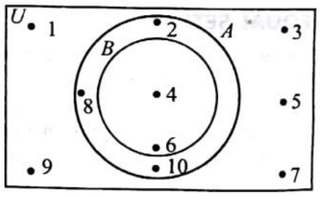
(ii) If $A = \{1, 2, 3, 4\}$, $\{2, 5, 6\}$, $\{1, 3, 7, 8, 9\}$ and $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are sets under consideration, then set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ can be consider as universal set because all other three sets are the subset of this set.

VENN DIAGRAMS

In order to illustrate universal sets, subsets and certain operations on sets in a clear and simple way, we make use of geometric figures. These figures are called Venn-Diagrams. In Venn Diagrams, a universal set is represented by a rectangle and any other set is represented by a circle.

In the Venn-diagrams, the elements of the sets are written in their respective circles.

In the Venn-diagram $U = \{1, 2, 3, ..., 10\}$ is the universal set of which $A = \{2, 4, 6, 8, 10\}$ and $B = \{4, 6\}$ are subsets, and also $B \subset A$.



OPERATION ON SETS

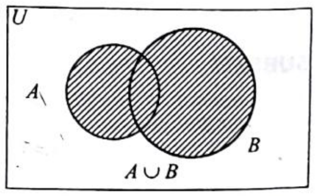
UNION OF SETS

The union of two sets A and B is the set which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write

$A \cup B = \{ x : x \in A \text{ or } x \in B \}$

The union of two sets can be represented by a Venn diagram as shown in Figure.

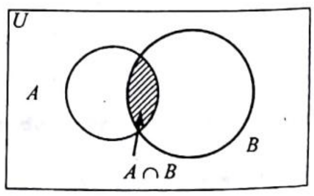
The shaded portion in Figure represents $A \cup B$.



INTERSECTION OF SETS

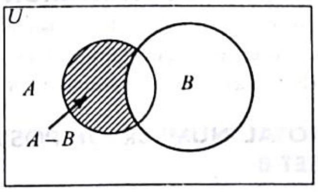
The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$

The shaded portion in Figure indicates the intersection of A and B.



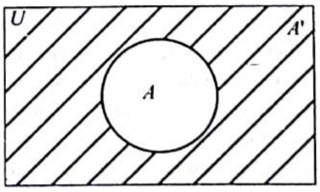
DIFFERENCE OF SETS

The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write $A - B$ and read as "A minus B".
Using the setbuilder notation, we can rewrite the definition of difference as
$A - B = \{ x : x \in A \text{ but } x \notin B \}$
The difference of two sets A and B can be represented by Venn diagram as shown in Figure.



COMPLEMENT OF A SET

Let U be the universal set and A be a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A' or A^c to denote the complement of a set A with respect to U. Thus, $A' = \{x : x \in U \text{ but } x \notin A \}$.
Obviously $A' = U - A$
The complement A' of a set A can be represented by a Venn diagram as shown in Figure
The shaded portion represents the complement of the set A.

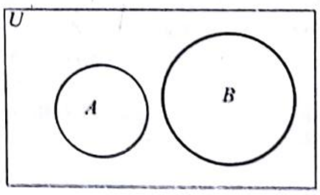


Some Properties of Complement of a Set

1. Complement laws: (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$
2. De Morgan's law: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
3. Law of double complementation : $(A')' = A$
4. Laws of empty set and universal set : $\phi' = U$ and $U' = \phi$.
   These laws can be verified by using Venn diagrams.

DISJOINT SETS

If A and B are two sets such that $A \cap B = \phi$, then A and B are called disjoint sets.
For example, let $A = \{ 2, 4, 6, 8 \}$ and $B = \{ 1, 3, 5, 7 \}$. Here A and B are disjoint sets, because there is no element common to both A and B. The disjoint sets can be represented by means of Venn diagram as shown in the Figure
In the adjoining diagram, A and B are disjoint sets.



CARDINAL NUMBER

Number of element in a set A is called cardinal number of set A.
It is represented by $n(A)$. If $A = \{a, b, c, d, e, f\}$, Then $n(A) = 6$
Theorem: If A and B are finite sets then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

RELATION

A relation R from a non-empty set A to non-empty set B is a set of ordered pairs (x, y) obtained by describing a rule which relate the elements x of set A to elements y of set B. Here y is called image of x under relation R.

A relation from a non-empty set A to non-empty set B is symbolically represented by $R : A \to B$.

Consider two non-empty sets:

$A = \{2, 3, 8\}$, $B = \{1, 2, 3, 4, 5\}$ and a rule "less than" which relate the elements of set A to the elements of set B. Then the relation from set A to B is $R = \{(2, 3), (2, 4), (2, 5), (3, 4), (3, 5)\}$

DOMAIN OF A RELATION

Domain of a relation is the set of all first elements of the ordered pairs in the relation.
So domain of the relation $\{(2,3), (2, 4), (2, 5), (3, 4), (3,5)\} = \{2, 3\}$

RANGE OF A RELATION

Range of a relation is the set of all second elements of the ordered pairs in the relation.
So range of the relation $\{(2,3), (2, 4), (2, 5), (3, 4), (3,5)\} = \{3, 4, 5\}$

CODOMAIN OF A RELATION

Codomain of a relation from a non-empty set A to non-empty set B is the set B.
Consider the relation R from set A to B where $A = \{2, 3, 8\}$ and $B = \{1, 2, 3, 4, 5\}$
Then codomain of the relation $R = B = \{1, 2, 3, 4, 5\}$

TOTAL NUMBER OF POSSIBLE RELATIONS FROM A NON-EMPTY SET A TO NON-EMPTY SET B

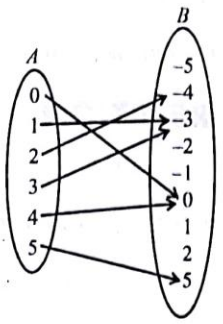
Actually each relation from a non-empty set A to non-empty set B is a sub set of cartesian product $A \times B$ of sets A and B.
Now number of elements in set $A \times B$
= (number of elements in set A) × (number of elements in set B)
If m and n are the number of elements in set A and set B respectively, then number of elements in $A \times B = m \times n$
Also we know that number of subsets of a set having p elements $= (2)^p$
So the total number of subsets of $A \times B = (2)^{m \times n}$
Since each subset of $A \times B$ is a relation from set A to B. Hence, Total number of relations from set A to set $B = (2)^{m \times n}$ where m and n are the number of elements of set A and set B respectively.

Note : If there is a relation from a non-empty set A to the same non-empty set A, then it said that 'relation in set A'. For example, a relation 'less than' in set $A = \{2, 3, 8\}$ is $\{(2, 3), (2, 8), (3, 8)\}$

FUNCTION

A function 'f' from a non-empty set A to a non-empty set B is symbolically written as $f: A \rightarrow B$, is a specific type of relation in which every element of set A related to an unique element of set B.
Since function is a special type of relation. So, it has also domain, codomain and range as of relation.
Like a relation, for the function $f: A \rightarrow B$, set A is called the domain and set B is called the codomain of the function f.
A real function is a function of which both domain and range are subsets of set of real numbers.

(i) Let us consider a relation from set A to B, where $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{-5, -4, -3, -2, 0, 1, 2, 5\}$ and the relation is defined by the equation $y = x^2 - 4x$.
Hence the relation $= \{(0, 0), (1, -3), (2, -4), (3, -3), (4, 0), (5, 5)\}$
This relation can be also shown as



- Here we see that each element of set A is related with unique (only one) element of set B. Hence the above relation is a function.

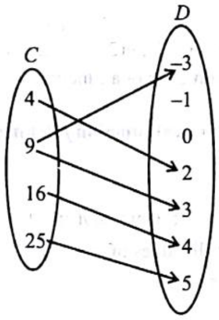
Domain of the function $= A$
$= \{0, 1, 2, 3, 4, 5\}$
Range of the function $= \{-4, -3, 0, 5\}$
Co-domain of the function $= B$
$= \{-5, -4, -3, -2, -1, 0, 1, 2, 5\}$

Since domain and range both are the subsets of set of real numbers, therefore the function is real function.

(ii) Let consider another relation from set C to D, where $C = \{4, 9, 16, 25\}$, $D = \{-3, -1, 0, 2, 3, 4, 5\}$ and the relation defined by the equation $y = \sqrt{x}$.

Hence the relation = $\{(4, 2), (9, -3), (9, 3), (16, 4), (25, 5)\}$

This relation can be also shown as



Here we see that an element 9 of set C is related with two elements –3 and 3 of set D. Hence each element of set C is not related with the unique (only one) element of set D.

Therefore the given relation is not a function.

MISCELLANEOUS

Solved Examples

Example 1 : **Let R be the relation in the set N given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.**

(a) $(2, 4) \in R$ **(b)** $(3, 8) \in R$ **(c)** $(6, 8) \in R$ **(d)** $(8, 7) \in R$

SOLUTION : (c)

Option (c) satisfies the condition that $a = b - 2$ i.e. $6 = 8 - 2$ and $b > 6$, i.e. $b = 8$

$\Rightarrow$ option (c) is correct.

Example 2 : **The domain of the function $f(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$ is**

(a) $(-\infty, 1)$ **(b)** $(-\infty, 1) \cup (2, \infty)$ **(c)** $(-\infty, 1] \cup [2, \infty)$ **(d)** $(2, \infty)$

SOLUTION : (b)

For $f(x)$ to be defined, we must have

$x^2 - 3x + 2 = (x - 1)(x - 2) > 0 \Rightarrow x < 1$ or > 2

Domain of $f = (-\infty, 1) \cup (2, \infty)$.

Example 3 : **The relation f is defined by** $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$

Show that f is a function and g is not a function.

SOLUTION :

(i) $f(x) = x^2$, is defined in the interval $0 \le x \le 3$

Also, $f(x) = 3x$ is defined in the interval $3 \le x \le 10$

At $x = 3$, from $f(x) = x^2, f(3) = 3^2 = 9$ and from $f(x) = 3x, f(3) = 3 \times 3 = 9$

$\therefore$ f is defined at $x = 3$, Hence, f is function.

(ii) $g(x) = x^2$ is defined in the interval $0 \le x \le 2$

$g(x) = 3x$ is defined in the interval $2 \le x \le 10$

But at $x = 2$, $g(x) = x^2$, $g(2) = 2^2 = 4$ and from $g(x) = 3x$, $g(2) = 3 \times 2 = 6$

At $x = 2$, relation on g has two values. $\therefore$ Relation g is not a function.

Example 4 : **Find the domain and the range of the real function f defined by $f(x) = \sqrt{x-1}$**

SOLUTION :

(i) $f(x) = \sqrt{x-1}$, f is not defined for $x - 1 < 0$ or $x < 1$; Domain $f(x) = \{x : x \in R, x \ge 1\}$

(ii) Let $f(x) = y = \sqrt{x-1}$ $\quad\therefore$ y is well defined for all values of $x \ge 1$.

$\therefore$ Range $= \{y : y \in R\} - \{y : y \in R \text{ and } -1 < y < 1\}$

Example 5 : **Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function from Z to Z defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.**

SOLUTION : We have $f(x) = ax + b$, for $x = 1, f(x) = 1$,

$\therefore$ $a + b = 1$ ... (i)

for $x = 2, f(x) = 3$, $\therefore$ $3 = a \times 2 + b$ or $2a + b = 3$ ... (ii)

subtracting Eq. (ii) from (i) $a = 2$, $b = -1$

Example 6 : **Let $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$ Are the following true?**

(i) f is a relation from A to B **(ii) f is a function from A to B**

Justify your answer in each case.

SOLUTION :

(i) f is a subset of $A \times B$. $\therefore$ f is a relation from A to B

(ii) The element $2 \in A$, has two images 9 and 11

$\therefore$ f is not a function from A to B

Example 7 : **Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A to B.**

SOLUTION : $A = \{x, y, z\}$, $B = \{1, 2\}$

Now, $n(A \times B) = n(A).\, n(B) = 6$ $\therefore$ The number of subset of $n(A \times B)$ is $2^6 = 64$

So, the number of relations from A into B is 64.

Example 8 : **Find the range of each of the following function.**

(i) $f(x) = 2 - 3x, x \in R, x > 0$ **(ii) $f(x) = x^2 + 2$, x is a real number**

(iii) $f(x) = x$, x is a real number

SOLUTION :

(i) Let $f(x) = y = 2 - 3x$

Now $x > 0$, $\therefore$ $y < 2$, $\therefore$ Range $(f) = \{y : y \in R \text{ and } y < 2\}$

(ii) Let $f(x) = y = x^2 + 2$, since x is a real number $\Rightarrow$ $y \ge 2$

$\therefore$ Range $(f) = \{y : y \in R \text{ and } y \ge 2\}$

(iii) Let $f(x) = y = x$, since, x is a real number; $\Rightarrow$ y is also a real number.

$\therefore$ Range $(f) = \{y : y \in R\}$

Example 9 : $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation R from A to B by $R = \{(x, y)$: the difference between x and y is odd: $x \in A, y \in B\}$. Write R in roster form.

SOLUTION : $R = \{(x, y)$: The difference between x and y is odd, $x \in A, y \in B\}$
Where, $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$
$R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

Example 10 : Write the relation $R = \{(x, x^3) : x$ is a prime number less than 10$\}$ in roster form.

SOLUTION : Prime numbers less than 10 are 2, 3, 5, 7. Hence $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

Example 11 : Which of the following relations are functions ? Give reasons. If it is a function determine its domain and range.
(i) $\{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$
(ii) $\{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}$
(iii) $\{(1, 3), (1, 5), (2, 5)\}$

SOLUTION :

(i) We have $f = \{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$
No two ordered pairs have the same first component, therefore this relation is a function.
Domain $(f) = \{2, 5, 8, 11, 14, 17\}$
Range $(f) = \{1\}$

(ii) We have $f = \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}$
We observe that no two ordered pairs have the same first component so, f is a function.
Domain $(f) = \{2, 4, 6, 8, 10, 12, 14\}$; Range $(f) = \{1, 2, 3, 4, 5, 6, 7\}$

(iii) $f = \{(1, 3), (1, 5), (2, 5)\}$
We observe that 1 has appeared more than once as first component of the ordered pairs in f.
Therefore, f is not a function.

Example 12 : Find the domain and range of two values of $f(x) = -|x|$.

SOLUTION : $f(x) = -|x|, f(x) \leq 0, \forall x \in R$

Domain of $f = R$; Range of $f = \{y : y \in R, y \leq 0\}$

Example 13 : Write the set $\{x : x$ is a positive integer and $x^2 < 40\}$ in the roster form.

SOLUTION : The required numbers are 1, 2, 3, 4, 5, 6. So, the given set in the roster form is $\{1, 2, 3, 4, 5, 6\}$.

Example 14 : Write the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$ in the set-builder form.

SOLUTION : We see that each member in the given set has the natural number numerator one less than the denominator. Also, the numerator begin from 1 and do not exceed 6. Hence, in the set-builder form the given set can be written as

$$\left\{x : x = \frac{n}{n+1}, \text{ where } n \text{ is a natural number and } 1 \leq n \leq 6\right\}$$

Example 15 : Which of the following pairs of sets are equal? Justify your answer.
(i) X, the set of letters in "ALLOY" and B, the set of letters in "LOYAL".
(ii) $A = \{n : n \in Z$ and $n^2 \leq 4\}$ and $B = \{x : x \in R$ and $x^2 - 3x + 2 = 0\}$.

SOLUTION:

(i) We have, $X = \{A, L, L, O, Y\}$, $B = \{L, O, Y, A, L\}$. Then X and B are equal sets as repetition of elements in a set do not change a set. Thus, $X = \{A, L, O, Y\} = B$

(ii) $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2\}$. Since $0 \in A$ and $0 \notin B$ therefore A and B are not equal sets.

Example 16 : **Are the following pair of sets equal ? Give reason.**

(i) $A = \{2, 3\}$, $B = \{x : x$ **is a solution of** $x^2 + 5x + 6 = 0\}$

(ii) $A = \{x : x$ **is a letter in the word FOLLOW**$\}$
$B = \{y : y$ **is a letter in the word WOLF**$\}$.

SOLUTION : We have,

(i) $A = \{2, 3\}$, $B = \{x : x$ is a solution of $x^2 + 5x + 6 = 0\}$

Now, $x^2 + 5x + 6 = 0$

$\Rightarrow x^2 + 3x + 2x + 6 = 0$

$\Rightarrow x(x + 3) + 2(x + 3) = 0$

$\Rightarrow (x + 3)(x + 2) = 0 \Rightarrow x = -2, -3$

Therefore, $B = \{-2, -3\}$

Here, we observe that the elements of set A are not exactly the same to that of set B, hence A and B are not equal sets.

(ii) We have, $A = \{x : x$ is a letter in the word FOLLOW$\}$

$\Rightarrow A = \{F, O, L, W\}$

And $B = \{y : y$ is a letter in the word WOLF$\}$

$\Rightarrow B = \{W, O, L, F\}$

Here, we observe that the elements of both sets are exactly same, hence the sets are equal.

Example 17 : **Write the following intervals in set builder form:**

(i) $(-3, 0)$ **(ii)** $[6, 12]$ **(iii)** $(6, 12]$ **(iv)** $[-23, 5)$

SOLUTION : The following intervals are written in set builder form as :

(i) $(-3, 0)$ is an open interval which does not include both -3 and 0. So it can be shown in the set builder form of a set as : $\{x : x \in R, -3 < x < 10\}$

(ii) $[6, 12]$ is a closed interval which includes both 6 and 12. So it can be shown in the set builder form as $\{x : x \in R, 6 \le x \le 10\}$

(iii) $(6, 12]$ is an interval closed at the second end only i.e. it excludes 6 but includes 12. So it is shown in the set builder form as : $\{x : x \in R, 6 < x \le 12\}$

(iv) $[-23, 5)$ is an interval closed at the first end point but open at the second end point. It means that the interval includes -23 but excludes 5. It is written in the set builder form as : $\{x : x \in R, -23 \le x < 5\}$.

Example 18 : **What universal set would you propose for each of the following :**

(i) **The set of right triangles.** **(ii)** **The set of isosceles triangles.**

SOLUTION :

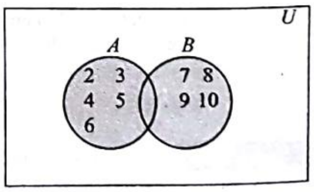
(i) The universal set for the set of right triangles is the set of triangles.

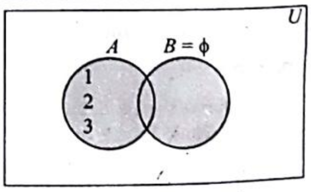
(ii) The universal set for the set of isosceles triangle is the set of equilateral triangles or set of triangles.

Example 19 : **Find the union of each of the following pairs of sets :**

(i) $A = \{x : x$ **is a natural number and** $1 < x \le 6\}$
$B = \{x : x$ **is a natural number and** $6 < x \le 10\}$

(ii) $A = \{1, 2, 3\}$, $B = \phi$





SOLUTION :

(i) $A = \{x : x$ is a natural number and $1 < x \le 6\}$

$\Rightarrow A = \{2, 3, 4, 5, 6\}$

$B = \{x : x$ is a natural number and $6 < x \le 10\}$

$\Rightarrow B = \{7, 8, 9, 10\}$

$\therefore A \cup B = \{2, 3, 4, 5, 6\} \cup \{7, 8, 9, 10\}$

$\Rightarrow A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(ii) We have, $A = \{1, 2, 3\}$, $B = \phi$

$\Rightarrow A \cup B = \{1, 2, 3\} \cup \phi$

$\Rightarrow A \cup B = \{1, 2, 3\}$

Example 20 : **If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$, then find $(A \cap B)$.**

SOLUTION : Let $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow$ x is a multiple of 3 and x is a multiple of 4.

$\Leftrightarrow$ x is a multiple of 3 and 4 both

$\Leftrightarrow$ x is a multiple of 12

$\Leftrightarrow$ $x = 12n,\ n \in Z$

Hence $A \cap B = \{x : x = 12n, n \in Z\}$

Example 21 : **If X and Y are two sets such that $X \cup Y$ has 50 elements, X has 28 elements and Y has 32 elements, how many elements does $X \cap Y$ have ?**

SOLUTION : Given that $n(X \cup Y) = 50$, $n(X) = 28$, $n(Y) = 32$, $n(X \cap Y) = ?$

By using the formula, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$,

we find that $n(X \cap Y) = n(X) + n(Y) - n(X \cup Y) = 28 + 32 - 50 = 10$

Example 22 : **In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football ?**

SOLUTION :

Let X be the set of students who like to play cricket and Y be the set of students who like to play football. Then $X \cup Y$ is the set of students who like to play at least one of the two games, and $X \cap Y$ is the set of students who like to play both games.

Given $n(X) = 24$, $n(Y) = 16$, $n(X \cup Y) = 35$, $n(X \cap Y) = ?$

Using the formula $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$, we get.

$$35 = 24 + 16 - n(X \cap Y)$$

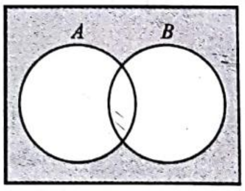
Thus, $n(X \cap Y) = 5$ i.e., 5 students like to play both games.

Example 23 : **Draw the appropriate Venn diagram for each of the following**

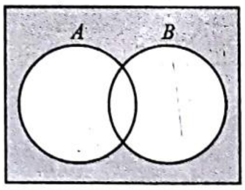
(i) $(A \cup B)'$ **(ii)** $A' \cap B'$ **(iii)** $(A \cap B)'$ **(iv)** $A' \cup B'$

SOLUTION :

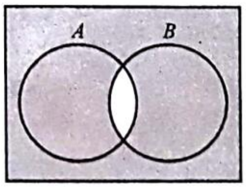
(i) $(A \cup B)'$ is represented by shaded area



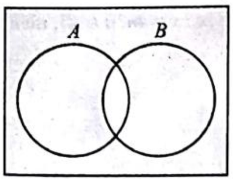
(ii) $A' \cap B'$ is represented by shaded Area



(iii) $(A \cap B)'$ is represented by the shaded area



(iv) $A' \cup B'$ is represented by shaded area



Example 24 : **Decide, among the following sets which sets are subsets of one and another.**
$A = \{x : x \in R$ **and** x **satisfy** $x^2 - 8x + 12 = 0\}$, $B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\ldots\ldots\}$, $D = \{6\}$

SOLUTION : We have $A = \{2, 6\}$,(since, only $x = 2$, and $x = 6$, satisfy the equation $x^2 - 8x + 12 = 0$)
$B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\ldots\ldots\}$ and $D = \{6\}$
Every element of A is in B and C; $\therefore A \subset B$ and $A \subset C$
Again every element of B is in C; $\therefore B \subset C$
Also, every element of D is in A, B and C ; $\therefore D \subset A, D \subset B$, and $D \subset C$.

Example 25 : **In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.**

(i) **If $x \in A$ and $A \in B$, then $x \in B$**
(ii) **If $A \subset B$ and $B \in C$, then $A \in C$**
(iii) **If $A \subset B$ and $B \subset C$, then $A \subset C$**
(iv) **If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$**
(v) **If $x \in A$ and $A \not\subset B$, then $x \in B$**
(vi) **If $A \subset B$ and $x \notin B$, then $x \notin A$**

SOLUTION :

(i) **False;** Let $A = \{1\}$, $B = \{\{1\}, 2\}$
$\therefore$ $1 \in A$ and $A \in B$ but $1 \notin B$
So, $x \in A$ and $A \in B$ need not imply that $x \in B$

(ii) **False;** Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{\{1, 2\}, 3\}$
$\therefore$ $A \subset B$ and $B \in C$ but $A \notin C$
Thus, $A \subset B$ and $B \in C$ need not imply that $A \notin C$.

(iii) **True;** Let $x \in A$, Then $A \subset B \Rightarrow x \in B$, $B \subset C \Rightarrow x \in C$
Thus, $A \subset B$ and $B \subset C \Rightarrow A \subset C$

(iv) **False;** Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 5\}$
Then $A \not\subset B$ and $B \not\subset C$ But $A \subset C$
Thus, $A \not\subset B$ and $B \not\subset C$ need not imply that $A \not\subset C$

(v) **False;** Let $A = \{1, 2\}$ and $B = \{2, 3, 4, 5\}$
Then $1 \in A$ and $A \not\subset B$ as $1 \not\subset B$
Thus, $x \in A$ and $A \not\subset B$ need not imply that $x \in B$

(vi) **True;** Let $A \subset B$, then $x \in A \Rightarrow x \in B \Leftrightarrow x \notin B \Rightarrow x \notin A$

EXERCISE 1

FIB Fill in the Blanks

DIRECTIONS: *Complete the following statements with an appropriate word / term to be filled in the blank space(s).*

1. A set which does not contain any element is called the __________ set.
2. Every set is a __________ of itself.
3. The__________ set is a subset of every set.
4. The collection of all subsets of a set A is called the __________set of A.
5. A relation R from a non-empty set A to non-empty set B is symbolically represented by __________.
6. If A and B are two non-empty sets then $(A \cup B)'$ = __________.
7. For non-empty set A, we have $(A')'$ = __________
8. Number of element in a set A is called __________ of set A.

T/F True /False

DIRECTIONS: *Read the following statements and write your answer as true or false.*

1. If A and B are two sets such that $A \cap B = \phi$ then A and B are called empty sets.
2. $(A \cap B)' = A \cup B'$ for any sets A and B.
3. If A is the set of points on a line then A is infinite set.
4. If B is the set of the days of the week then B is a finite set.
5. A real function is a function in which both domain and range are subsets of set of real numbers.
6. If A is any set and U is an universal set then $A \cup A' = \phi$ and $A \cap A' = U$
7. If $A = \{a, e, i, o, u\}$ then $n(A) = 5$
8. A set, consisting of a single element is called a void set.

MTC Match the Columns

DIRECTIONS: *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D, ...) in Column I have to be matched with statements (p, q, r, s, ...) in Column II.*

1.

Column I	Column II
(A) Domain of a relation is	(p) the set of all elements of U which are not the elements of that set.
(B) Range of a relation is	(q) the set of all those elements which belong to both sets.
(C) Co-domain of a relation is	(r) the set of all first elements of the ordered pairs in the relation.
(D) The complement of a set is	(s) the set of all second elements of the ordered pairs in the relation.
(E) The intersection of two sets is	(t) from a non-empty set A to non-empty set B is the set B.

2.

Column I	Column II
(A) If $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$ then $A \cup B =$	(p) $\{2, 4\}$
(B) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 9, 12\}$ then $A \cap B =$	(q) $\{9\}$
(C) The set $\{5\}$ is a	(r) $\{1, 2, 3, 5, 7\}$
(D) $\{x \in N : 5 < x < 6\}$ is an	(s) $\{1,3\}$
(E) If $A=\{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 6, 7, 8, 9\}$ then $A - B$ is	(t) singleton set
(F) If $A=\{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 6, 7, 8, 9\}$ then $B - A$ is	(u) empty set

VSAQ Very Short Answer Questions

DIRECTIONS: *Give answer in one word or one sentence.*

1. What can you say about $n(A \cup B)$, when A and B are disjoint sets?
2. If $A = \{2, 3, 4, 5, 6, 7\}$ and $B = \{3, 5, 7, 9, 11, 13\}$ then find $A - B$ and $B - A$.
3. Let the set of natural numbers be the universal set and let $A = \{2, 4, 6, 8, ...\}$. Find A'.
4. Define the "Union of Sets".
5. If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ be a relation from A to B then find (i) Dom (R) (ii) Range (R)
6. If $f(x) = 3x^4 - 5x^2 + 9$ then find $f(x-1)$.

7. Define a finite set.

8. Find the domain of the function $f(x)=\dfrac{x^2+3x+5}{x^2-5x+4}$

9. Find the domain of the function $f(x)=\dfrac{x^2+2x+1}{x^2-8x+12}$

10. Let $A = \{a, e, i, o, u\}$ and $B = \{a, i, u\}$. Show that $A \cup B = A$

11. How many elements has $P(A)$, if $A = \phi$?

12. Let $A = \{a, b\}, B = \{a, b, c\}$. Is $A \subset B$? What is $A \cup B$?

SAQ Short Answer Questions

DIRECTIONS: *Give answer in 2-3 sentences.*

1. If $y = f(x) = \dfrac{1-x}{1+x}$, prove that $x = f(y)$.

2. Define a relation R on the set N of natural numbers by $R = \{(x, y) : y = x + 5, x$ is a natural number less than $4 : x, y \in N\}$. Depict this relationship using roster form. Write down the domain and the range.

3. Let $A = \{1, 2, 3, 4, 6\}$. Let R be the relation on A defined by $\{(a, b) : a, b \in A, b$ is exactly divisible by $a\}$
   (i) Write R in roster form
   (ii) Find the domain of R
   (iii) Find the range of R.

4. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A to B.

5. Are the following pair of sets equal? Give reasons.
   (i) $A = \{2, 3\}, B = \{x : x$ is solution of $x^2 + 5x + 6 = 0\}$
   (ii) $A = \{x : x$ is a letter in the word FOLLOW$\}$
   $B = \{y : y$ is a letter in the word WOLF$\}$

6. If $A = \{3, 5, 7, 9, 11\}, B = \{7, 9, 11, 13\}, C = \{11, 13, 15\}$ and $D = \{15, 17\}$; find
   (i) $A \cap B$ (ii) $B \cap C$
   (iii) $A \cap C$ (iv) $A \cap (B \cup D)$

LAQ Long Answer Questions

DIRECTIONS: *Give answer in four to five sentences.*

1. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket ? How many like tennis?

2. In a group of 400 people, 250 can speak Hindi and 200 can speak English. How many can speak both Hindi and English?

3. Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$ and $B = \{3, 4, 5\}$. Find $A', B', A' \cap B', A \cup B$ and hence show that $(A \cup B)' = A' \cap B'$

4. Write the following sets in roster form :
   (i) $A = \{x : x$ is an integer and $-3 < x < 7\}$
   (ii) $B = \{x : x$ is a natural number less than $6\}$
   (iii) $C = \{x : x$ is a two-digit natural number such that the sum of its digits is $8\}$
   (iv) $D = \{x : x$ is a prime number which is divisor of $60\}$
   (v) $E =$ The set of all letters in the word TRIGONOMETRY
   (vi) F = The set of all letters in the word BETTER.

5. Write the following sets in the set-builder form :
   (i) $\{3, 6, 9, 12\}$
   (ii) $\{2, 4, 8, 16, 32\}$
   (iii) $\{5, 25, 125, 625\}$
   (iv) $\{2, 4, 6, ...\}$
   (v) $\{1, 4, 9, ..., 100\}$

6. Which of the following sets are finite or infinite
   (i) The set of months of a year
   (ii) $\{1, 2, 3, ...\}$
   (iii) $\{1, 2, 3, ..., 99, 100\}$
   (iv) The set of positive integers greater than 100
   (v) The set of prime numbers less than 90

EXERCISE 2

Multiple Choice Questions

DIRECTIONS: *This section contains multiple choice questions. Each questions has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

1. If $f(x)=\dfrac{x^2+1}{x-2}$ then value of $f(2)$

(a) 2 (b) 10
(c) 5 (d) does not exist

2. Range of the function $y=\dfrac{x^2}{1+x^2}$ is

(a) $(-1, 1)$ (b) $[0, 1)$
(c) $[1, 2]$ (d) $(-2, 2)$

3. Which of the following relation is a function ?

(a) $\{(a, b)\ (b, e)\ (c, e)\ (b, x)\}$
(b) $\{(a, d)\ (a, m)\ (b, e)\ (a, b)\}$
(c) $\{(a, d)\ (b, e)\ (c, d)\ (e, x)\}$
(d) $\{(a, d)\ (b, m)\ (b, y)\ (d, x)\}$

4. The domain of the function $f(x)=\dfrac{1}{\sqrt{x^2-3x+2}}$ is

(a) $(-\infty, 1)$ (b) $(-\infty, 1)\cup(2, \infty)$
(c) $(-\infty, 1)\cup(2, \infty)$ (d) $(2, \infty)$

5. If A and B are two sets, then $A\cap(A\cup B)'$ equals :

(a) A (b) B
(c) ϕ (d) None

6. Which of the following is a null set?

(a) $\{0\}$
(b) $\{x : x > 0 \text{ or } x < 0\}$
(c) $\{x : x^2 = 4 \text{ or } x = 3\}$
(d) $\{x : x^2 + 1 = 0, x \in R\}$

7. The set of intelligent students in a class is :

(a) A null set
(b) a singleton set
(c) a finite set
(d) Not a well defined collection

8. Consider the following equations :

1. $A - B = A - (A \cap B)$
2. $A = (A \cap B) \cup (A - B)$
3. $A - (B \cup C) = (A - B) \cup (A - C)$

Which of these is/are correct ?

(a) 1 and 3 (b) 2 only
(c) 2 and 3 (d) 1 and 2

9. If A is the set of the divisors of the number 15, B is the set of prime numbers smaller than 10 and C is the set of even numbers smaller than 9, then $(A \cup C) \cap B$ is the set

(a) $\{1, 3, 5\}$ (b) $\{1, 2, 3\}$
(c) $\{2, 3, 5\}$ (d) $\{2, 5\}$

10. Given the sets

$A=\{1, 3, 5\}$, $B=\{2, 4, 6\}$ and $C=\{0, 2, 4, 6, 8\}$. Which of the following may be considered as universal set for all the three sets A, B and C ?

(a) $\{0, 1, 2, 3, 4, 5, 6\}$
(b) ϕ
(c) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(d) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

More than One Correct

DIRECTIONS: *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which ONE OR MORE may be correct.*

1. Which of the following is/are finite set ?

(a) Set of even natural numbers less than 100.
(b) Set of all persons on the earth.
(c) Set of all lines in a plane.
(d) Set of even prime natural numbers.

2. Which of the following is/are infinite set ?

(a) $A=\{x : x \in z \text{ and } x^2-5x+6=0\}$
(b) $B=\{x : x \in z \text{ and } x^2 \text{ is even}\}$
(c) $C=\{x : x \in z \text{ and } x^2=36\}$
(d) $D=\{x : x \in z \text{ and } x>-10\}$

3. Which of the following is/are empty set ?

(a) Set of all even natural numbers divisible by 5.
(b) $\{x : x^2-2=0 \text{ and } x \text{ is rational}\}$
(c) $\{x : x$ is a natural number, $x < 8$ and simultaneously $x > 12\}$
(d) $\{x : x$ is a point common to any two parallel lines$\}$.

4. Which of the following is/are correct?

(a) A set is a well-defined collection of objects.
(b) A power set of a set A is collection of all subsets of A.
(c) For any two sets A and B, $(A\cup B)'=A'\cap B'$ and $(A\cap B)'=A'\cup B'$
(d) For any two sets A and B, $(A\cup B)'=A'\cup B'$ and $(A\cap B)'=A'\cap B'$

5. Which of the following is/are correct?
   (a) The domain of R is the set of all first elements of the ordered pairs in a relation R.
   (b) The domain of R is the set of all second elements of the ordered pairs in a relation R.
   (c) The range of the relation R is the set of all second elements of the ordered pairs in a relation R.
   (d) The range of the relation R is the set of all first elements of the ordered pairs in a relation R.

PBQ Passage Based Questions

DIRECTIONS: *Study the given paragraph(s) and answer the following questions.*

If $A = \{x : x \text{ is a natural number}\}$,
$B = \{x : x \text{ is an even natural number}\}$,
$C = \{x : x \text{ is an odd natural number}\}$
and $D = \{x : x \text{ is a prime number}\}$ then

1. $A \cap B$ is
   (a) B (b) C
   (c) D (d) ϕ
2. $A \cap C$ is
   (a) B (b) C
   (c) D (d) ϕ
3. $A \cap D$ is
   (a) B (b) C
   (c) D (d) ϕ

A&R Assertion & Reason

DIRECTIONS: *Each of these questions contains an Assertion followed by reason. Read them carefully and answer the question on the basis of following options. You have to select the one that best describes the two statements.*

(a) If both **Assertion** and **Reason** are **correct** and Reason is the **correct explanation** of Assertion.
(b) If both **Assertion** and **Reason** are correct, but Reason **is not the correct explanation** of Assertion.
(c) If **Assertion** is **correct** but **Reason** is **incorrect**.
(d) If **Assertion** is **incorrect** but **Reason** is **correct**.

1. **Assertion :** If $A = \{1, 2, 3, 4, 5, 6\}$ $B = \{7, 8, 9, 10, 11\}$ and $C = \{6, 8, 10, 12, 14\}$ then A and B are disjoint sets.
   Reason : Two sets A and B are said to be disjoint if $A \cap B = \phi$.
2. **Assertion :** Set $\{x : x \in R, 3 \le x \le 4\}$ can be written as [3, 4] in the interval form.
   Reason : If a, b are real numbers such that $a < b$, then the set $\{x : x \in R \text{ and } a \le x \le b\}$ is called the closed interval [a, b].
3. **Assertion :** The set of all rectangles is contained in the set of all squares.
   Reason : The sets $P = \{a\}$ and $B = \{\{a\}\}$ are equal.
4. **Assertion :** For any two sets A and B, we have $A - B = \{x : x \notin A \text{ and } x \in B\}$
   Reason : For any two sets A and B, we have $A - B = \{x : x \in A \text{ and } x \notin B\}$ and $B - A = \{x : x \in B \text{ and } x \notin A\}$
5. **Assertion :** If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ be a relation from A to B. Then,
   Dom $(R) = \{1, 3, 5\}$ and Range $(R) = \{8, 6, 2, 4\}$
   Reason : If R is a relation from set A to set B, then
   Domain $(R) = \{x : (x, y) \in R\}$
   Range $(R) = \{y : (x, y) \in R\}$

MMQ Multiple Matching Questions

DIRECTIONS : *Following question has four statements (A, B, C and D) given in Column I and statements (p, q, r, s,....) in Column II. Any given statement in Column I can have correct matching with one or more statement(s) given in Column II. Match the entries in column I with entries in column II.*

1.

Column I	Column II
(A) Open interval	(p) {2, 6, 10, 14} and {3, 7, 11, 15}
(B) Closed interval	(q) {5}
(C) Disjoint Set	(r) $\{x \in R : a < x < b\}$
(D) Singleton Set	(s) $\{x \in R : a \le x \le b\}$
	(t) {2, 6, 10} and {3, 7, 11}
	(u) (a, b) or $]a, b[$
	(v) $[a, b]$

SOLUTIONS

EXERCISE – I

FILL IN THE BLANKS

1. empty 2. subset 3. empty
4. Power. 5. $R : A \to B$ 6. $(A \cup B)' = A' \cap B'$
7. $(A')' = A$ 8. Cardinal number.

TRUE/FALSE

1. False. If $A \cap B = \phi$ then A and B are called disjoint sets.
2. True 3. True 4. True 5. True
6. False.
   $A \cup A' = U$ and $A \cap A' = \phi$
7. True.
   $n(A)$ = Number of element in set A
8. False.
   A set, consisting of a single element is called a singleton set.

MATCH THE COLUMNS

1. (A)$\to$ r, (B)$\to$ s, (C) $\to$ t, (D)$\to$ p, (E)$\to$ q
2. (A)$\to$ r, (B)$\to$ s, (C) $\to$ t, (D)$\to$ u, (E)$\to$ p, (F)$\to$ q

VERY SHORT ANSWER QUESTIONS

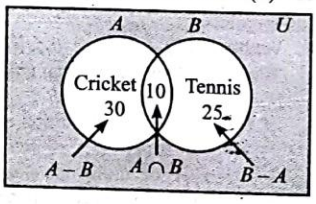
1. We know, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
   Since, A and B are disjoint sets
   $\therefore \quad A \cap B = \phi \Rightarrow n(A \cap B) = 0$
   $\therefore \quad n(A \cup B) = n(A) + n(B)$
2. $A - B = \{2, 4, 6\}, B - A = \{9, 11, 13\}$
3. $A' = \{1, 3, 5, \ldots\ldots\ldots\ldots\}$
4. Let A and B be two sets. The union of A and B is the set of all those elements which belong either to A or to B or to both A and B.
5. (i) Dom $(R) = \{1, 3, 5\}$ (ii) Range $(R) = \{8, 6, 2, 4\}$
6. $f(x-1) = 3(x-1)^4 - 5(x-1)^2 + 9$
   $$= 3x^4 - 12x^3 + 13x^2 - 2x + 7$$
7. A set consisting of a definite number of elements is called a finite set, otherwise the set is called an infinite set.
8. Since $x^2 - 5x + 4 = (x-4)(x-1)$, the function $f(x)$ is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is $R - \{1, 4\}$.
9. $$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12} = \frac{(x+1)^2}{(x-2)(x-6)}$$
   The function is not defined at $x = 2, 6$
   Domain of $f = \{x : x \in R \text{ and } x \neq 2, x \neq 6\}$
   $= R - \{2, 6\}$
10. We have, $A \cup B = \{a, e, i, o, u\} = A$.
    This example illustrates that union of sets A and its subset B is the set A itself, i.e., if $B \subset A$; then $A \cup B = A$.
11. If $A = \phi$, then by the definition of power set, we have
    $P(A) = P(\phi) = \{\phi\}$
    = a set containing one element.
12. Yes, $A \subset B$, because every element of A is also an element of B, therefore. A is a subset of B i.e., $A \subset B$.
    $A \cup B = \{a, b\} \cup \{a, b, c\} = \{a, b, c\}$

SHORT ANSWER QUESTIONS

1. Given, $y = f(x) = \dfrac{1-x}{1+x}$ ....(i)
   $$\text{Now, } f(y) = \frac{1-y}{1+y} = \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x$$
2. $R = \{(x, y) : x = y - 5, x$ is a natural number less than 4; $x, y \in N\} = \{(1, 6), (2, 7), (3, 8)\}$
   Domain = $\{1, 2, 3\}$; Range = $\{6, 7, 8\}$
3. (i) Given $A = \{1, 2, 3, 4, 6\}$ and R is relation on A defined by $\{(a, b) : a \in A, b \in A, a$ divides $b\}$
   Clearly, $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2)$ $(2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$,
   (ii) Domain $(R) = \{1, 2, 3, 4, 6\}$
   (iii) Range $(R) = \{1, 2, 3, 4, 6\}$
4. $A = \{x, y, z\}, B = \{1, 2\}, \quad A \times B = \{(x, 1), (x, 2), (y, 1)$ $(y, 2), (z, 1), (z, 2)\}$
   Now, $n(A \times B) = 6$, Therefore, the number of subsets of $n(A \times B) = 2^6 = 64$
   So, the number of relations from A to B is 64.
5. (i) No Since $x^2 + 5x + 6 = 0 \; \therefore \; (x + 2)(x + 3) = 0$
   $\Rightarrow x = -2$ or $x = -3$
   Solution set, $B = \{-2, -3\} \; \therefore A \neq B$
   (ii) Yes $A = \{x : x$ is a letter in the word FOLLOW$\}$
   $= \{F, O, L, W\}$
   $B = \{y : y$ is a letter in the word WOLF $\}$
   $B = \{W, O, L, F\}$
   Since, every element of A is in B and every element of B is in A.
   $\Rightarrow A = B$
6. (i) $A \cap B = \{3, 5, 7, 9, 11\} \cap \{7, 9, 11, 13\} = \{7, 9, 11\}$
   (ii) $B \cap C = \{7, 9, 11, 13\} \cap \{11, 13, 15\} = \{11, 13\}$
   (iii) $A \cap C = \{3, 5, 7, 9, 11\} \cap \{11, 13, 15\} = \{11\}$
   (iv) $A \cap (B \cup D) = \{3, 5, 7, 9, 11\} \cap (\{7, 9, 11, 13\} \cup \{15, 17$
   $= \{3, 5, 7, 9, 11\} \cap \{7, 9, 11, 13, 15, 17\} = \{7, 9, 11\}$

LONG ANSWER QUESTIONS

1. Let A be the set of people who like cricket and B be the set of people who like tennis. Then $n(A \cup B) = 65$, $n(A) = 40$, $n(A \cap B) = 10$
We know that,
$n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow 65 = 40 + n(B) - 10$
$\Rightarrow n(B) = 35$
So, 35 people like tennis



Number of people who like tennis only and not cricket
$= n(B - A)$
Also,
$n(B) = n(B - A) + n(A \cap B)$
$35 = n(B - A) + 10$
$n(B - A) = 35 - 10 = 25$

2. Let H denote the set of peoples who speak Hindi and E denote the set of peoples who speak English. We are given that,
$n(H) = 250$, $n(E) = 200$, $n(H \cup E) = 400$
We know that,
$n(H \cup E) = n(H) + n(E) - n(H \cap E)$
$\Rightarrow 400 = 250 + 200 - n(H \cap E)$
$\Rightarrow n(H \cap E) = 450 - 400 = 50$
Hence, 50 people can speak Hindi as well as English.

3. We have $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$,
$A' = U - A = \{1,4,5,6\}$
and $U = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 5\}$ and
$B' = U - B = \{1, 2, 6\}$
$\therefore A' \cap B' = \{1,6\}$ ...(i)
Also $A \cup B = \{2,3,4,5\}$
$(A \cup B)' = U - (A \cup B) = \{1,6\}$ ... (ii)
From (i) and (ii), $A' \cap B' = (A \cup B)'$. Hence, proved.

4. (i) $A = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
(ii) $B = \{1, 2, 3, 4, 5\}$
(iii) $C = \{17, 26, 35, 44, 53, 62, 71, 80\}$
(iv) $D = \{2, 3, 5\}$
(v) E = {T, R, I, G, O, N, M, E, Y}
(vi) F = {B, E, T, R}

5. (i) $\{x : x$ is a natural number multiple of 3 and $x < 15\}$
(ii) $\{x : x = 2^n, n \in N$ and $n < 6\}$
(iii) $\{x : x = 5^n$, and $n \in N$ and $n \leq 4\}$
(iv) $\{x : x$ is an even natural number $\}$
(v) $\{x : x = n^2, n \in N$ and $n < 11\}$

6. (i) The set of months of a year.
It is a finite set as there are 12 members of the set which are the months of the year
(ii) $\{1, 2, 3, ...\}$. It is an inifinte set since there are infinite number of natural numbers.
(iii) $\{1, 2, 3, ...99, 100\}$: It is a finite set it contains, first 100 natural numbers.
(iv) The set of positive integers greater than 100. It is an infinite set since there are infinite number of positive integers viz : 101, 102, 103 ....>100
(v) The set of prime numbers less than 99. This is a finite set because the set is {2, 3, 5, 7, ...97}.

EXERCISE – 2

MULTIPLE CHOICE QUESTIONS

1. (d) $f(2) = \dfrac{2^2+1}{2-2} = \dfrac{5}{0}$ which does not exist.

2. (b) Clearly, y is defined for all real values of x.
$\therefore$ Domain of $y = (-\infty, \infty)$
We have $y = \dfrac{x^2}{1+x^2} \Rightarrow x^2 y + y = x^2$
$\Rightarrow x^2 = \dfrac{y}{1-y}$ i. e., $x = \sqrt{\dfrac{y}{1-y}} = \dfrac{\sqrt{y(1-y)}}{(1-y)}$
For x to be real, $1 - y \neq 0$ i.e. $y \neq 1$ and $y(1-y) \geq 0$
$\Rightarrow y(y-1) \leq 0$ and $y \neq 1 \Rightarrow 0 \leq y < 1$
Range of $y = [0, 1)$.

3. (c) Since in (c) each element is associated with unique element while in (a) element b is associated with two elements, in (b) element a is associated with three elements and in (d) element b is associated with two elements so. (c) is function.

4. (b) For $f(x)$ to be defined, we must have
$x^2 - 3x + 2 = (x-1)(x-2) > 0 \Rightarrow x < 1$ or > 2
Domain of $f = (-\infty, 1) \cup (2, \infty)$.

5. (c) $A \cap (A' \cup B)' = A \cap (A' \cap B')$
$= (A \cap A') \cap B' = \phi \cap B' = \phi$

6. (d) $x^2 + 1 = 0$ has no solution in R

7. (d)

8. (d) Check by creating Venn-diagram.

9. (a) $A = \{1, 3, 5, 15\}$, $B = \{2, 3, 5, 7\}$ $C = \{2, 4, 6, 8\}$
$\therefore A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 15\}$
$(A \cup C) \cap B = \{2,3,5\}$

10. (c)

MORE THAN ONE CORRECT

1. (a, b, d)
2. (b, d)
(a) $A = \{2, 3\}$
(b) $B = \{..........., -6, -4, -2, 0, 2, 4, 6, ...\}$
(c) $C = \{6, -6\}$
(d) $D = \{-9, -8, -7, ...\}$.
3. (b, c, d) 4. (a, b, c) 5. (a, c)

PASSAGE BASED QUESTIONS

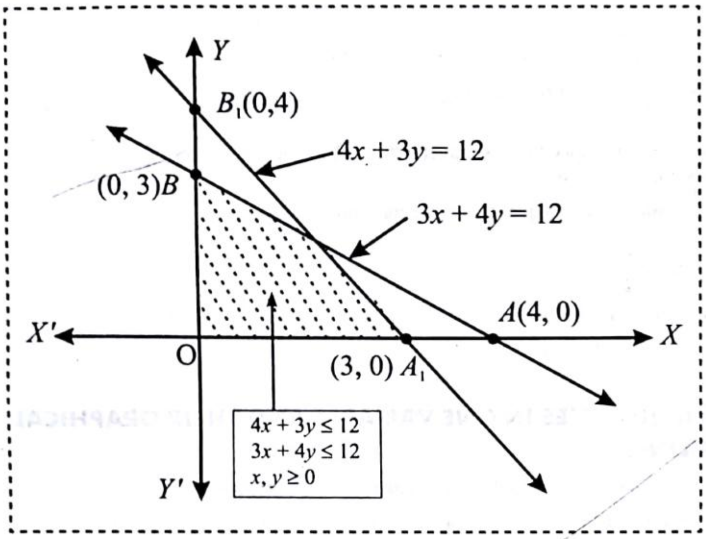
1. (1) (a) $A \cap B = B$ $(\because B \subset A)$
(2) (b) $A \cap C = C$ $(\because C \subset A)$
(3) (c) $A \cap D = D$ $(\because D \subset A)$

ASSERTION & REASON

1. (a) Assertion : $A \cap B = \phi$.
2. (a) Both statements are correct.
3. (c) Assertion is true but reason is false.
In rectangles if length becomes equal to breadth then rectangles becomes squares.
4. (d) Assertion is false.
Reason is true.
5. (a)

MULTIPLE MATCHING QUESTIONS

1. (A)→ r, u; (B)→ s, v; (C)→ p, t; (D)→ q



12

CHAPTER

Linear Inequalities

INTRODUCTION

When we compare two quantities in day-to-day situations, it is more likely that the two quantities are unequal rather than equal. A student may not write a three hour examination paper in exactly three hours. He may finish it in less time or more time. It rarely happens as planned.

The distance shown on a railway ticket from one station to other is either little less or a little more. Thus, inreality inequalities occur quite frequently in practical life, so it is natural to expect that their study is important in mathematics too.

INEQUALITY

Two numbers or algebraic expressions related by a symbol $>$, $<$, $\geq$ or $\leq$ is called an inequality.

For examples : $5 < 7$, $x \geq 3$, $x < 5$, $3 < x \leq 8$, $ax + b > 0$, $2x + y \geq 0$ are inequality.

Remember :

(i) Sign of inequality does not change when equal numbers added to (or subtracted from) both sides of an inequality.
Example : If $x \geq 3$, then $x + 2 \geq 5$. Also if $y < 8$, then $y - 2 < 6$

(ii) Sign of inequality does not change when both sides of an inequality can be multiplied (or divided) by the same positive number.
Example : If $x > -4$ then $2x > -8$. Also if $y \leq 8$, then $\frac{y}{2} \leq 4$

(iii) But when both sides are multiplied or divided by a negative number, then the sign of inequality is reversed.
Example : If $x > 3$, then $-2x < -6$. Also, if $x \leq 12$, then $-\frac{x}{3} \geq -4$

ALGEBRAIC SOLUTION OF LINEAR INEQUALITIES IN ONE VARIABLE AND THEIR GRAPHICAL REPRESENTATION ON THE NUMBER LINE:

(i) Any solution of an equality in one variable is the values of the variable which make it a true statement.

(ii) To represent the solution $x < a$ (or $x > a$) on a number line, put a very small blank circle on the number a and dark the line to the left (or right) of the number a. In the equality $x < a$ or $x > a$ the point $x = a$ on the number line is not included in the solutions of the inequality.

(iii) To represent the solution $x \leq a$ (or $x \geq a$) on a number line, put a very small filled dark circle on the number a and dark the line to the left (or right) of the number a.
In the inequality $x \leq a$ or $x \geq a$, the point $x = a$ on the number line is included in the solutions.

Illustration 1 : Solve the inequality $3(x - 1) \geq 2(x - 3)$ and represent the solution on the number line

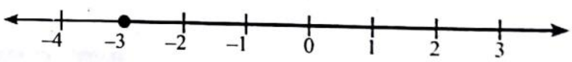
SOLUTION : $3(x - 1) \geq 2(x - 3)$

$\Rightarrow \quad 3x - 3 \geq 2x - 6 \qquad ...(i)$

Subtracting $2x$ from both sides of inquality '$\geq$' of (i), we get

$3x - 3 - 2x \geq 2x - 6 - 2x$

$\Rightarrow \quad x - 3 \geq -6 \qquad ...(ii)$



Add 3 on both sides inequality
'$\geq$' of (ii), we get

$x - 3 + 3 \geq -6 + 3$

$x \geq -3$

Illustration 2 : Solve the inequality $\frac{2x}{3} < \frac{x}{2} + 1$ and represent the solution on the number line.

SOLUTION : $\frac{2x}{3} < \frac{x}{2} + 1 \qquad ...(i)$

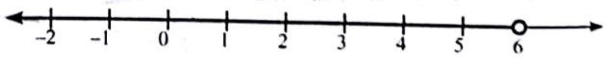
'$<$' of (i), we get

$$\frac{2x}{3} - \frac{x}{2} < \frac{x}{2} + 1 - \frac{x}{2}$$

$\Rightarrow \quad \frac{2x}{3} - \frac{x}{2} < 1 \qquad ...(ii)$

$\Rightarrow$ Multiply by 6 (L.C.M. of 3 and 2) on both sides of inequality '$<$' of (ii), we get

$$6 \times \frac{2x}{3} - 6 \times \frac{x}{2} < 6$$



$\Rightarrow \quad 4x - 3x < 6$

$\Rightarrow \quad x < 6$

Illustration 3 : Solve the inequality $-\frac{x}{3} - 5x > 8$ and repersent the solution on the number line

SOLUTION : $-\frac{x}{3} - 5x > 8$ ...(i)

Multiply by –3 on both sides of inequality '>' of equation (i), we get

$$(-3) \times \left(-\frac{x}{3}\right) - (-3) \times 5x < (-3) \times 8$$

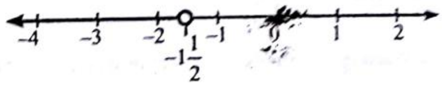
$\Rightarrow$ $x + 15x < -24$

$\Rightarrow$ $16x < -24$ ...(ii)

Divide by 16 on both sides of inequality '<' of (ii), we get

$$\frac{16x}{16} < \frac{-24}{16}$$

$\Rightarrow$ $x < -\frac{3}{2}$, $\Rightarrow$ $x < -1\frac{1}{2}$



ALGEBRAIC SOLUTION OF A SYSTEM OF LINEAR INEQUALITIES IN ONE VARIABLE

The common algebraic solution of all the solutions of individual inequality in one variable of the system is the algebraic solutions of the given system of linear inequalities in one variable.

Illustration 4 : Solve the following system of inequalities: $6x - 1 > 5x + 2$, $\frac{x}{2} + x \le \frac{x}{4} + 10$ and repersent the solution on the number line

SOLUTION : $6x - 1 > 5x + 2$ ...(i)

Subtract $5x$ from both sides of inequality '>' of equation (i), we get

$6x - 1 - 5x > 5x + 2 - 5x$

$\Rightarrow$ $x - 1 > 2$ ...(ii)

Add 1 on both sides of inequality '>' of (ii), we get

$x - 1 + 1 > 2 + 1$

$\Rightarrow$ $x > 3$ ...(iii)

Now $\frac{x}{2} + x \le \frac{x}{4} + 10$ ...(iv)

Subtract $\frac{x}{4}$ from both side of inequality '≤' of (iv), we get

$$\frac{x}{2} + x - \frac{x}{4} \le \frac{x}{4} + 10 - \frac{x}{4}$$

$\Rightarrow$ $\frac{x}{2} + x - \frac{x}{4} \le 10$ ...(v)

Multiply by 4 (L.C.M of 2 and 4) on both sides of inequality '≤' of (v), we get

$$4 \times \frac{x}{2} + 4x - 4 \times \frac{x}{4} \le 4 \times 10$$

$\Rightarrow$ $2x + 4x - x \le 40$

$\Rightarrow$ $5x \le 40$ ...(vi)

Divide by 5 on both sides of inequality '≤' of (v), we get

$$\frac{5x}{5} \le \frac{40}{5}$$

$\Rightarrow$ $x \le 8$ ...(vii)

From (iii) and (vii), we get

$3 < x \leq 8$

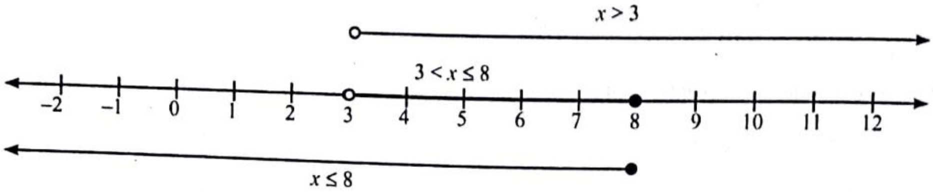


Illustration 5 : Solve the following system of inequalities $\frac{3x}{2} + 5 < \frac{x}{2}$, $x - \frac{5x}{3} > -8$

SOLUTION : $\frac{3x}{2} + 5 < \frac{x}{2}$ ...(i)

Subtracting $\frac{x}{2}$ from both sides of inequality '<' of (i), we get

$\frac{3x}{2} + 5 - \frac{x}{2} < \frac{x}{2} - \frac{x}{2}$

$\Rightarrow \quad \frac{3x}{2} - \frac{x}{2} + 5 < 0$ ...(ii)

Subtracting 5 from both sides of inequality '<' of (ii), we get

$\frac{3x}{2} - \frac{x}{2} + 5 - 5 < 0 - 5$

$\Rightarrow \quad \frac{3x}{2} - \frac{x}{2} < -5$ ...(iii)

Multiply by 2 (L.C.M. of 2 and 2) on both sides of inequality '<' of (iii), we get

$\frac{3x}{2} \times 2 - \frac{x}{2} \times 2 < -5 \times 2$

$\Rightarrow \quad 3x - x < -10$

$\Rightarrow \quad 2x < -10$ ...(iv)

Divide by 2 on both sides of inequality '<' of (iv), we get

$x < -\frac{10}{2}$

$\Rightarrow \quad x < -5$ ...(v)

And $x - \frac{5x}{3} > -8$ ...(vi)

Multiply by 3 on both sides of inequality '>' of (vi), we get

$3x - 3 \times \frac{5x}{3} > 3 \times (-8)$

$\Rightarrow \quad 3x - 5x > -24$

$\Rightarrow \quad -2x > -24$ ...(vii)

Divide by –2 on both sides of inequality '>' of (vii), we get

$\frac{-2x}{-2} > \frac{-24}{-2}$

$\Rightarrow \quad x > 12$ ...(viii)

There is no value of x which satisfies (v) and (viii) i.e., there is no common solution of the given two inequalities. Hence there is no solution of the given system of inequalities.

GRAPH OF A LINEAR EQUATION

(i) To draw the graph of a linear equation containing both x and y, find the point on the x-axis (by putting $y = 0$ in the equation) and on the y-axis (by putting $x = 0$ in the equation). Draw a line by joining these two points. This line is the graph of the given linear equation.

(ii) The graph of the equation $x = a$ (or $y = b$) is a straight line perpendicular to the x-axis at $x = a$ (or y-axis at $y = b$) where 'a' and 'b' are any real number.

Illustration 6: Draw the graph of $2x - y + 1 = 0$

SOLUTION : $2x - y + 1 = 0$ ...(i)

Put $y = 0$ in equation (i), we get

$$2x = -1 \Rightarrow x = -\frac{1}{2}$$

Put $x = 0$ in equation (i), we get

$$2 \times 0 - y + 1 = 0 \Rightarrow y = 1$$

x	$-\frac{1}{2}$	0
y	0	1

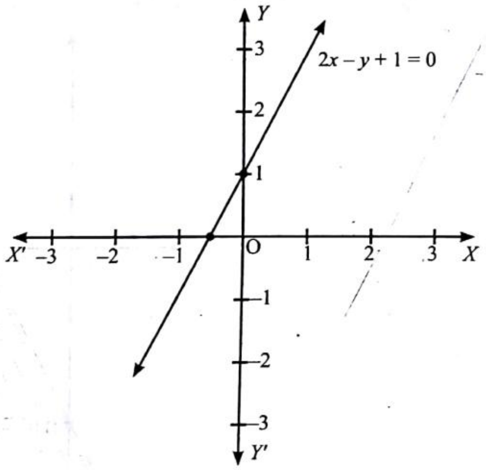
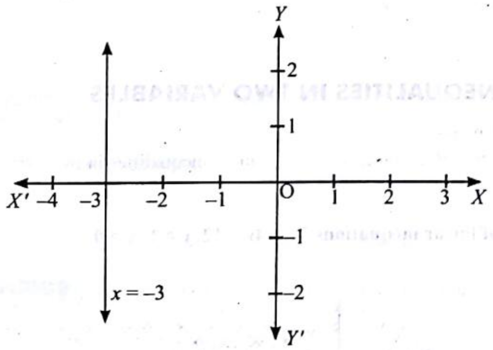


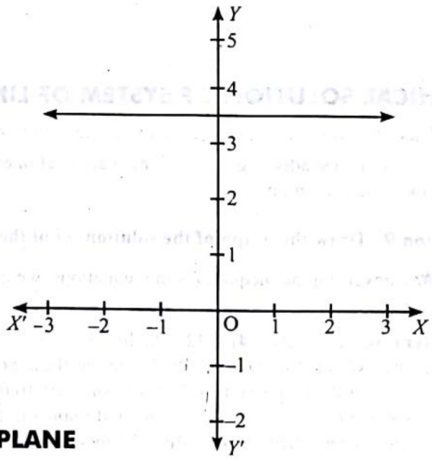
Illustration 7 : Draw the graph of (i) $x = -3$ and $y = \frac{7}{2}$

SOLUTION :

(i)



(ii)



TO FIND THE GRAPHICAL SOLUTION OF A LINEAR INEQUALITY IN TWO VARIABLES IN A CARTESIAN PLANE

(i) Draw the graph of the linear equation in two variables corresponding to the linear inequalities in two variables.

(ii) The line drawn in step (i) divide the cartesian plane in two half planes.

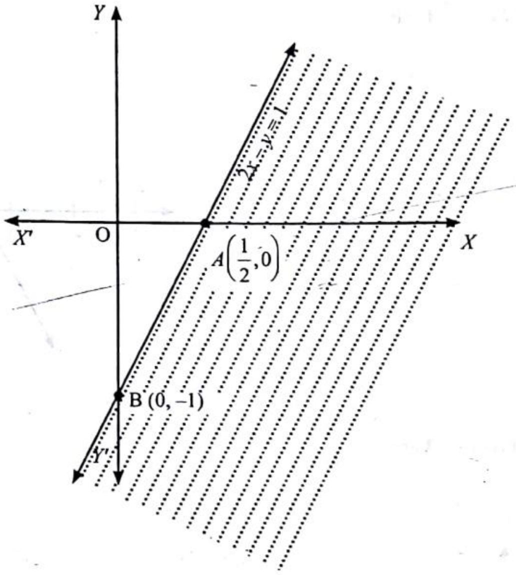
(iii) Take any point on either side of the line drawn in step (i), if co-ordinate of this point satisfy the linear inequality, then the half plane containing this point is the solution region of the inequality of the form $ax + by > c$ or $ax + by < c$

Hence to show the solution region of this form of inequality we shade the solution region. The point on the line $ax + by = c$ are not included in the solution region. So, the line drawn in step (i) should be broken or dotted, which means line is not included in the solution region.

If the inequality of the form $ax + by \geq c$ or $ax + by \leq c$ then the solution region is the half plane satisfy $ax + by > c$ or $ax + by < c$ containing all the points on the line $ax + by = c$. Hence to show the solution region of this form of inequality, we shade and dark the line drawn in step (i), which shows the solution region is the shaded half plane containing the line.

Illustration 8 : Solve the following inequations graphically $2x - y \geq 1$

SOLUTION : Converting the given inequation into equation, we obtain $2x - y = 1$. This line meets x and y-axes at $A(1/2, 0)$ and $B(0,-1)$ respectively. Joining these points by a thick line we obtain the line passing through A and B as shown in Fig. This line divides the xy-plane into two regions viz. one lying above it and the other lying below it. Consider the point $O(0,0)$. Clearly, (0,0) does not satisfy the inequation $2x - y \geq 1$. So, the region not containing the origin is represented by the given inequation as shown in Fig. Clearly it represents the solution set of the given inequation.



GRAPHICAL SOLUTION OF SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES

(i) Shade the solution region of each linear inequality in different ways.

(ii) The common shaded region of all the regions of linear inequalities of the given system of linear inequalities in two variables is the solution region.

Illustration 9 : Draw the graph of the solution set of the system of linear inequations $3x + 4y \geq 12, y \geq 1, x \geq 0$

SOLUTION: Converting the inequations into equations, we get

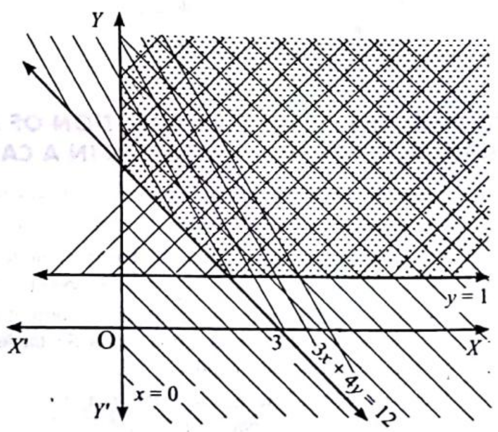
$3x + 4y = 12, y = 1, x = 0$

Region Represented by $3x + 4y \geq 12$: The line $3x + 4y = 12$ meets the coordinate axes at $A(0, 4)$ and $B(0, 3)$ joining these points by a thick line we get the graph of $3x + 4y = 12$. Since (0, 0) does not satisfy the inequation $3x + 4y \geq 12$. So, the half plane on the right side of the line repersented by $3x + 4y = 12$ including the line is represented by the inequation $3x + 4y \geq 12$.

Region Represented by $y \geq 1$: The line $y = 1$ is parallel to x-axis at a unit distance from it. Since (0, 0) does not satisfy the inequation $y \geq 1$. So, the region lying above and including the line $y = 1$ is represented by $y \geq 1$.

Region Represented by $x \geq 0$: Clearly, $x \geq 0$ represents the region lying on the right side of y-axis including y-axis.

The solution set of the given system of linear equations is the intersection of the above regions as shown in Figure.



MISCELLANEOUS

Solved Examples

Example 1 : **Solve the inequation $12+1\frac{5}{6}x \le 5+3x,\ x \in R$. Represent the solution set on a number line.**

SOLUTION : We have $12+1\frac{5}{6}x \le 5+3x \Rightarrow 12+\frac{11}{6}x \le 5+3x$

$\Rightarrow$ $72 + 11x \le 30 + 18$ [Multiplying both sides by 6]

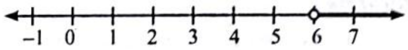
$\Rightarrow$ $11x \le 18x - 42$ [Adding –72 on both sides]

$\Rightarrow$ $-7x \le -42$ [Adding $-18x$ on both sides]

$\Rightarrow$ $x \ge 6$ [Dividing both sides by –7]

$\therefore$ Solution set = $\{x : x \ge 6,\ x \in R\}$

This set can be represented on the number line, as shown below :



Example 2 : **Solve the following inequalities:**

(a) $2x+1>\frac{3}{2}x-2$ **(b)** $x+1>x+2$ **(c)** $2x+2>2x+1$.

SOLUTION :

(a) $2x+1>\frac{3}{2}x-2 \Leftrightarrow \frac{1}{2}x>-3 \Leftrightarrow x>-6$

(b) $x+1>x+2 \Leftrightarrow 0.x>1$, consequently, the given inequality has no solution.

(c) $2x+2>2x+1 \Leftrightarrow 0.\ x>-1$, consequently, the given inequality is satisfied by any real number, i.e., the set of solutions is set R of all real numbers.

Example 3 : **Solve the inequality $\frac{x+1}{x-2}>1$.**

SOLUTION : The given inequality is equivalent to the inequality

$$\frac{x+1}{x-2}-1>0\text{, i.e., }\frac{3}{x-2}>0,$$

and therefore any number $x>2$ is a solution and there are no other solutions.

Example 4 : **Solve the following system of linear inequalities : $x + 2 > 11$; $2x \le 20$**

SOLUTION : The given system of inequalities is

$x+2>11$ .... (1)

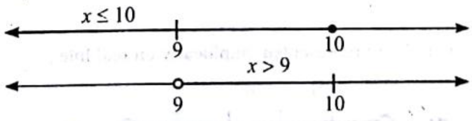
$2x \le 20$ .... (2)

Now, $x+2>11 \Rightarrow x>11-2 \Rightarrow x>9 \Rightarrow x \in (9, \infty)$

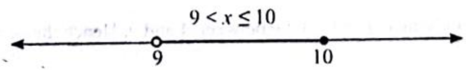
$\therefore$ Solution of inequality (1) is $x>9$ .... (3)

and $2x \le 20 \Rightarrow x \le 10 \Rightarrow x \in (-\infty, 10]$

$\therefore$ Solution of inequality (2) is $x \le 10$ .... (4)



Hence, the solution of the given system is given by $9<x \le 10 \Rightarrow x \in (9, 10]$



Example 5 : Solve $-8 \le 5x - 3 < 7$

SOLUTION : In this case, we have two inequalities, $-8 \le 5x - 3$ and $5x - 3 < 7$, which we will solve simultaneously.

We have $-8 \le 5x - 3 < 7$ or $-5 \le 5x < 10$ or $-1 \le x < 2$

Example 6 : Solve $-5 \le \dfrac{5-3x}{2} \le 8$

SOLUTION : We have $-5 \le \dfrac{5-3x}{2} \le 8$

or $-10 \le 5 - 3x \le 16$ or $-15 \le -3x \le 11$ or $5 \ge x \ge -\dfrac{11}{3}$, which can be written as $-\dfrac{11}{3} \le x \le 5$.

Example 7 : **Find the solution of the following system of linear inequalities**

$\dfrac{4x}{3} - \dfrac{9}{4} < x + \dfrac{3}{4}$, $\dfrac{7x-1}{3} - \dfrac{7x+2}{6} > x$

SOLUTION :

The given system of linear inequalities is

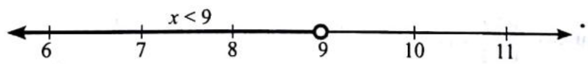
$$\frac{4x}{3} - \frac{9}{4} < x + \frac{3}{4} \qquad \text{...... (1)}$$

$$\frac{7x-1}{3} - \frac{7x+2}{6} > x \qquad \text{...... (2)}$$

From inequality (1), we have

$$\frac{16x-27}{12} < \frac{4x+3}{4} \Rightarrow 16x - 27 < 12x + 9$$

$\Rightarrow$ $4x < 36$ $\Rightarrow$ $x < 9$



Thus solution of inequality (1) is given by

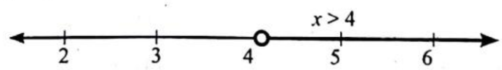
$x < 9$ .... (3)

$\Rightarrow$ $x \in (-\infty, 9)$

From inequality (2), we get

$$\frac{14x-2}{6} - \frac{7x+2}{6} > x \Rightarrow 14x - 2 - 7x - 2 > 6x$$

$\Rightarrow$ $7x - 4 > 6x$ $\Rightarrow x > 4$ $\Rightarrow x \in (4, \infty)$

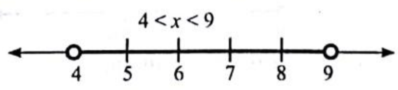


Thus solution of inequality (2) is given by

$x > 4$ .... (4)

$\Rightarrow$ $x \in (4, \infty)$

The solution set of inequality (1) and (2) are represented graphically on real line :



Clearly the common values of x satisfying (3) and (4), lie between 4 and 9. Hence the solution of the given system is given by $4 < x < 9$ $\Rightarrow x \in (4, 9)$

Example 8 : **Ravi obtained 70 and 75 marks in first two unit test. Find the number if minimum marks he should get in the third test to have an average of at least 60 marks.**

SOLUTION : Let Ravi get x marks is third unit test

$\therefore$ Average marks obtained by Ravi $= \dfrac{70+75+x}{3}$

He is obtained atleast 60 marks

$$\frac{70+75+x}{3} \geq 60 \Rightarrow \frac{145+x}{3} \geq 60$$

Multiplying by 3, $145+x \geq 60\times 3 = 180$

Transposing 145 to RHS $\Rightarrow x \geq 180-145 = 35$

$\therefore$ Ravi should got atleast 35 marks in the third unit test.

Example 9 : **A man wants to cut three lengths from a single piece of board of length 91 cm, the second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second ?**

SOLUTION : Let x be the length of the shortest board, then $x + 3$ is the second length and $2x$ is the third length. Thus,

$x+(x+3)+2x \leq 91 \Rightarrow 4x+3 \leq 91 \Rightarrow 4x \leq 88 \Rightarrow 4x \leq 22$

According to the problem,

$2x \geq (x+3)+5 \Rightarrow x \geq 8$

At least 8 cm long but not more than 22 cm long.

Example 10 : **Solve the inequalities $-3x + 2y \geq -6$ graphically.**

SOLUTION

Let us draw the line $-3x+2y=-6$

The line passes through $A(2, 0)$, $B(0, -3)$

The line AB represents the equation $-3x+2y=-6$

Now, consider the inequality

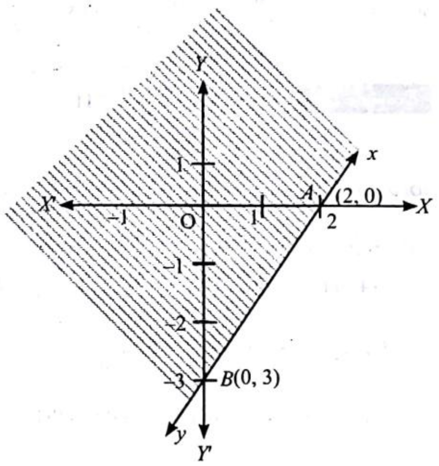
$-3x+2y \geq -6$

Putting $x=0, y=0$

$-0+0 \geq -6$

Which is true

Hence, the solution region is the half plane above and including the line repersented by equation $-3x + 2y = -6$. Graph of this inequality is shown in the figure by shaded area.



Example 11: **Solve the system of inequalities graphically : $3x + 2y \leq 12, x \geq 1, y \geq 2$**

SOLUTION

Inequalities are $3x+2y \leq 12, x \geq 1, y \geq 2$

(i) The line $3x+2y=12$ passes through A(4, 0), B(0, 6)

AB represents the line

$3x+2y=12$

Putting $x=0, y=0$ in $3x+2y \leq 12$

$-0+0 \leq 12$

Which is true

$\therefore$ Origin lies in the region $3x+2y \leq 12$

i.e $3x+2y \leq 12$ represents the region below the line $3x+2y=12$ including the line.

(ii) The line $x = 1$ passes through $E(1, 0)$ and $F(1, 2)$. This line is represented by EF.

Consider the inequality $x \geq 1$

Putting $x = 0$, we get $0 \geq 1$, which is not true

Hence, origin does not lie in the region $x \geq 1$

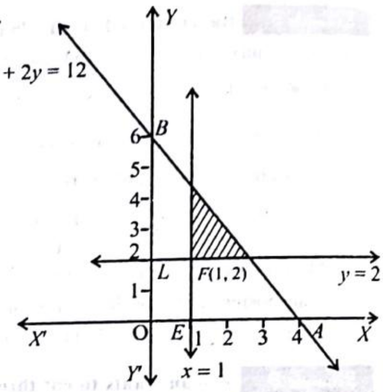
Hence, $x \geq 1$ repersents the region on the right of EF including the line EF.

(iii) The line $y = 2$, passes through the points $F(1, 2)$ and L (0, 2)

Putting $y = 0$ in $y \geq 2$, we get $0 \geq 2$, which is false.

Origin does not lie in the region. Hence, $y \geq 2$ is represented by the region above the line repersented by $y = 2$ including the line. Hence, the region satisfying the system of inequalities.

$3x + 2y \leq 12, x \geq 1, y \geq 2$ is the shaded region shown in the figure.



Example 12 : Solve: $-12 < 4 - \frac{3x}{-5} \leq 2$

SOLUTION : Subtracting 4, we get

$$-12-4 < 4-\frac{3x}{-5}-4 \leq 2-4 \Rightarrow -16 < \frac{3x}{5} \leq -2$$

Multiplying by $\frac{5}{3}$, $-16\times\frac{5}{3} < \frac{3x}{5}\times\frac{5}{3} \leq -2\times\frac{5}{3} \Rightarrow \frac{-80}{3} < x \leq \frac{-10}{3}$

or $\quad x \in \left[\frac{-80}{3}, \frac{-10}{3}\right]$ i.e. $3 > \frac{-80}{3}$ and $x \leq -\frac{10}{3}$

Example 13 : Solve : $7 \leq \frac{3x+11}{2} \leq 11$

SOLUTION : Multiplying by 2 we get $7\times 2 \leq \frac{3x+11}{2}\times 2 \leq 11\times 2 \Rightarrow 14 \leq 3x+11 \leq 22$

Subtracting by 11

$$14-11 \leq 3x+11-11 \leq 22-11 \Rightarrow 3 \leq 3x \leq 11$$

Dividing by 3, $1 \leq x \leq \frac{11}{3}$ or $x \in \left[1, \frac{11}{3}\right]$

i.e., x is greater than or equal to 1 and less than or equal to $\frac{11}{3}$

EXERCISE 1

Fill in the Blanks

DIRECTIONS: *Complete the following statements with an appropriate word / term to be filled in the blank space(s).*

1. The two signs > and < are called the signs of ....
2. > denotes ....
3. < denotes ....
4. ≥ denotes ....
5. ≤ denotes ....

True / False

DIRECTIONS: *Read the following statements and write your answer as true or false.*

1. The square of any real number is always greater than or equal to 0.
2. If $a > b$ then $a + c > b + c$
3. If $a < b$ and $b < c$ then we can write $a < b < c$.
4. $5x^2 + 6 > 7$ and $6x^3 + 6y^3 \le 8$ are some of the examples for linear inequations.
5. If $x + y \le 5$ then either $x \le 5$ or $y \le 5$ or both
6. If $a > b$, then $\frac{a}{c} < \frac{b}{c}$ for all a, b and $c \in R$ where $c < 0$

Match the Columns

DIRECTIONS: *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D, E) in Column I have to be matched with statements (p, q, r, s, t) in column II.*

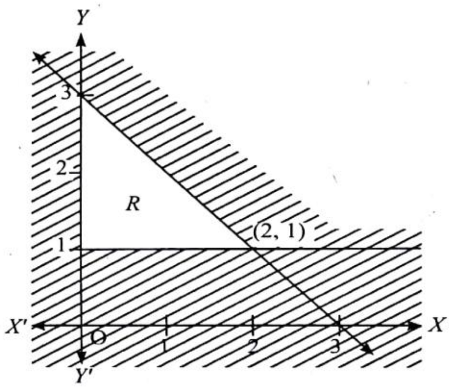
1.

Column I (Inequalities)	Column II (Solution Set)
(A) $-11 \le 4x - 3 \le 13$	(p) $[2, 4]$
(B) $3x - 6 \ge 0$ and $4x - 10 \le 6$	(q) $(-\infty, -3]$
(C) $4x - 12 \ge 0$	(r) $(3, \infty)$
(D) $7x + 9 > 30$	(s) $[3, \infty)$
(E) $3x + 17 \le 2(1 - x)$	(t) $[-2, 4]$

Very Short Answer Questions

DIRECTIONS: *Give answer in one word or one sentence.*

1. Define the term "Inequation".
2. Solve : $30x < 200$ when 'x' is a natural number.
3. Solve : $7x + 3 < 5x + 9$. Show the graph of the solutions on number line.
4. Find the solution set for the inequation $\frac{1}{x+1} > 0$.
5. In the diagram below, the unshaded region R is defined by three inequalities, one of which is $x \ge 0$. Write down the other two inequalities.



6. Solve : $2 \le 3x - 4 \le 5$

Short Answer Question

DIRECTIONS: *Give answer in 2-3 sentences.*

1. Solve the inequalities $3(x-1) \le 2(x-3)$ for real x.
2. Solve : $3x + 8 > 2$, when
   (i) x is an integer
   (ii) x is a real number
3. To receive grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examination are 87, 92, 94 and 95. Find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.
4. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.
5. Solve the inequalities $\frac{x}{3} > \frac{x}{2} + 1$ for real x.
6. Solve the inequalities $37 - (3x + 5) \ge 9x - 8(x - 3)$ for real x.

7. Solve the inequalities $3x+4y \le 12$ graphically in two-dimensional plane

8. Solve : $6 \le 3-(2x-4) < 12$

9. Solve: $-3 \le 4-\frac{7x}{2} \le 18$

10. Solve: $-15 < \frac{3(x-2)}{5} \le 0$

Long Answer Questions

DIRECTIONS: *Give answer in four to five sentences.*

1. Solve the following inequation graphically : $2x+3y \le 6$
2. Solve the following inequation graphically : $y \le -3$
3. Exhibit graphically the solution set of the linear inequations $3x+4y \le 12, 4x+3y \le 12, x \ge 0, y \ge 0$
4. Solve the following system of linear inequalities : $x+2 > 11; 2x \le 20$
5. Solve the system of inequalities graphically : $2x-y > 1$, $x-2y < -1$.

EXERCISE 2

Multiple Choice Questions

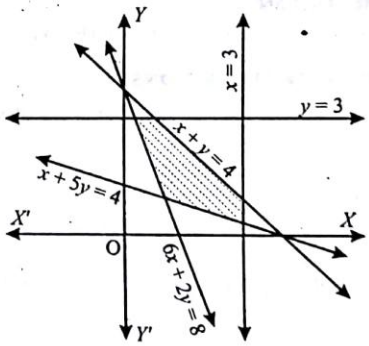
DIRECTIONS: *This section contains multiple choice questions. Each questions has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

1. The smallest value of x such that $-n \le x-4\sqrt{n}$, where n is a non-negative number, is
   (a) 2 (b) 0
   (c) 4 (d) 16
2. If $\frac{x+3}{x-3} < 1$, then which of the following cannot be the value of x ?
   (a) 0 (b) 1
   (c) 2 (d) 4
3. If $0 < \frac{2x-5}{2} < 7$ and x is an integer, then the sum of the greatest and least value of x is
   (a) 9 (b) 10
   (c) 6 (d) 12
4. The common solution set of the inequations $5 \le 2x+7 \le 8$ and $7 \le 3x+5 \le 9$ is
   (a) $\frac{2}{3} \le x \le \frac{4}{3}$ (b) $-1 \le x \le \frac{4}{3}$
   (c) $\frac{2}{3} \le x \le \frac{1}{2}$ (d) Null set
5. Set of solution for :
   $(x+5)-7(x-2) \ge 4x+9$ and
   $2(x-3)-7(x+5) \le 3x-9$ is
   (a) $[-4, 1]$ (b) $[1, 1]$
   (c) $[2, 1]$ (d) $[-3, 1]$
6. If x and y are any two positive real numbers then $x > y$ implies,
   (a) $-x < -y$ (b) $-x > -y$
   (c) $\frac{1}{x} > \frac{1}{y}$ (d) $\frac{-1}{x} > \frac{1}{y}$

Hots Subjective Questions

DIRECTIONS: *Answer the following questions.*

1. Solve the following inequations graphically:
   (i) $|x| \le 3$ (ii) $|y-x| \le 3$
   (iii) $|x-y| \ge 1$
2. Find the linear inequations for which the solution set is the shaded region given in Fig.



3. A company manufactures cassettes and its cost and revenue functions for a week are $C = 300+\frac{3}{2}x$ and $R = 2x$ respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold for the company to realize a profit ?
4. Solve the inequation:
   $\left|\frac{2}{x-4}\right| > 1, x \ne 4$
5. Solve : $\frac{|x|-1}{|x|-2} \ge 0$, $x \in R, x \ne \pm 2$

SOLUTIONS

EXERCISE I

FILL IN THE BLANKS

1. inequalities
2. greater than
3. less than
4. greater than or equal to
5. less than or equal to

TRUE/FALSE

1. True
2. True
3. True
4. False
5. True
6. True

MATCH THE COLUMNS

1. (A) $\to t$, (B) $\to p$, (C) $\to s$, (D) $\to r$, (E) $\to q$

VERY SHORT ANSWER QUESTIONS

1. An open sentence which consists of one of the symbols $>, <, \geq, \leq$ is called an inequation.

2. $30x < 200$

$\Rightarrow \dfrac{30x}{30} < \dfrac{200}{30} \Rightarrow x < \dfrac{20}{3}$

Since, x is a natural number

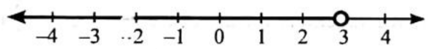
$\therefore$ the following values of x make the statement true.

1, 2, 3, 4, 5, 6

3. $7x + 3 < 5x + 9$.

or $2x < 6$ or $x < 3$

The graphical representation is



4. Given inequation is

$\dfrac{1}{x+1} > 0.$

$\Rightarrow \quad x > -1$

Solution Set $= \{x : x > -1\}$

5. Other two inequalities are $x + y \leq 3$ and $y \geq 1$.

6. Adding 4, we get

$2+4 \leq 3x-4+4 \leq 5+4 \Rightarrow 6 \leq 3x \leq 9$

Dividing by 3, $2 \leq x \leq 3$, Thus the solution is [2, 3]

SHORT ANSWER QUESTIONS

1. The inequality is $3(x-1) \leq 2(x-3)$

$\Rightarrow \quad 3x - 3 \leq 2x - 6$

Transposing $2x$ to L.H.S and -3 to R.H.S.

$\Rightarrow \quad 3x - 2x \leq -6 + 3$, $x \leq -3$,

$\therefore$ The solution is $(-\infty, -3]$

2. Inequality is $3x + 8 > 2$

Transposing 8 to RHS we get $3x > 2 - 8 = -6$

Dividing by 3, $x > -2$

(i) When x is an integer the solution is $-1, 0, 1, 2, 3$

(ii) When x is real, the solution is $(-2, \infty)$.

3. Let Sunita obtain x marks in the fourth examination.

$\therefore$ Average of marks of 5 examinations.

$= \dfrac{87+92+94+95+x}{5} = \dfrac{368+x}{5}$

This average must be atleast equal to 90

$\Rightarrow \quad \dfrac{368}{5} + x \geq 90$

$\Rightarrow \quad 368 + x \geq 1 = 5 \times 90 = 450 \Rightarrow x \geq 450 - 368 = 82$

i.e. She should obtain atleast 82 marks in the fifth examination.

4. Let shortest side measure x cm. Therefore the longest side will be $3x$ cm and third side will be $(3x - 2)$ cm

According to the problem,

$x + 3x + 3x - 2 \geq 61 \Rightarrow 7x - 2 \geq 61$

or $\quad 7x \geq 63 \Rightarrow x \geq 9$cm

Hence, the minimum length of the shortest side is 9 cm and the other sides measure 27 cm and 25 cm.

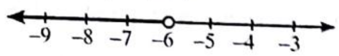
5. The inequality $\dfrac{x}{3} > \dfrac{x}{2} + 1$

Transposing $\dfrac{x}{2}$ to LHS.

Simplifying, $\dfrac{x}{3} - \dfrac{x}{2} > 1 \Rightarrow \dfrac{2x-3x}{6} > 1$ or $\dfrac{-x}{6} > 1$

Multiply by -6, $x < -6$

$\therefore$ The solution is $(-\infty, -6)$



6. The inequality is $37 - (3x + 5) \geq 9x - 8(x - 3)$

Simplifying $37 - 3x - 5 \geq 9x - 8x + 24$

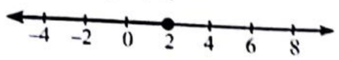
$\Rightarrow 32 - 3x \geq x + 24$

Transposing x to L.H.S and 32 to R.H.S, $-3x - x \geq 24 - 32$

or $-4x \geq -8$

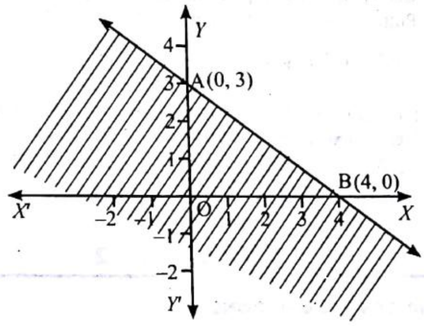
Dividing by -4, we get $x \leq 2$

$\therefore$ The solution is $(-\infty, 2]$



7. We draw the graph of the equation $3x + 4y = 12$. The line passes through the points (4, 0), (0, 3). This line is represented by AB.

Now consider the inequality $3x + 4y \le 12$



Putting $x = 0, y = 0$, we get

$0 + 0 = 0 \le 12$

Which is true

$\therefore$ Origin lies in the region of $3x + 4y \le 12$

The shaded region represents this inequality.

8. Dividing by 3, $2 \le -(2x-4) \le 4 \Rightarrow 2 \le -2x+4 \le 4$

Subtract 4, we get $\quad 2-4 \le -2x+4-4 \le 4-4$

or $\quad -2 \le -2x \le 0 \quad \Rightarrow \quad 1 \ge x \ge 0$ ($\because$ dividing by -2)

Hence, x is less than or equal to '1' and is greater than 0 i.e. $x \in [0, 1]$

9. Subtract 4, we get

$$-3-4 \le 4-\frac{7x}{2}-4 \le 18-4 \Rightarrow -7 \le -\frac{7x}{2} \le 14$$

Dividing by $\frac{-7}{2}$, $-7 \times \frac{-2}{7} \ge -\frac{7x}{2} \times \frac{-2}{7} \ge 14 \times \frac{-2}{7}$

$\Rightarrow \quad 2 \ge x \ge -4 \quad \Rightarrow \quad x \in [-4, 2]$

$\therefore \quad x$ is less than or equal to 2 and greater than or equal to -4.

10. Multiplying by 5,

$$-15 \times 5 < \frac{3(x-2)}{5} \times 5 \le 0 \times 5 \Rightarrow -75 < 3(x-2) \le 0$$

$\Rightarrow \quad \frac{-75}{3} < x-2 \le \frac{0}{3} \qquad$ (Divide by 3)

or $\quad -25 < x-2 \le 0$

Adding 2, we get

$$-25+2 < x-2+2 \le 0+2$$

$$\Rightarrow -23 < x \le 2$$

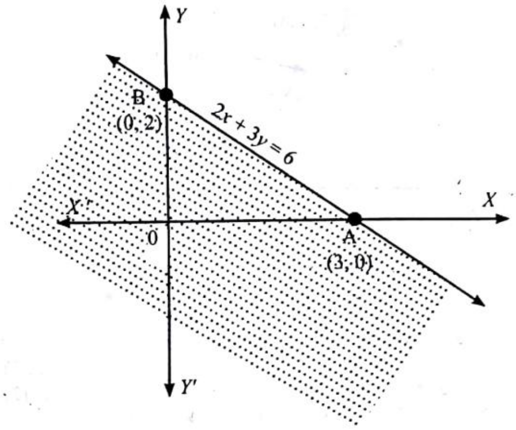
Hence, x is less than or equal to 2 and greater than -23 i.e. $x \in (-23, 2]$

LONG ANSWER QUESTIONS

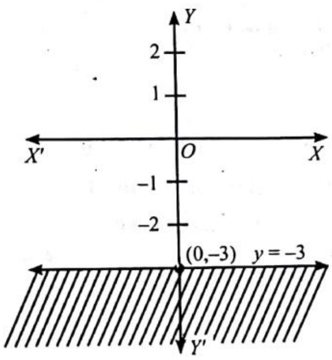
1. Given inequation can be rewritten as $2x + 3y = 6$.

By putting $y = 0$ and $x = 0$ in equation, we get $x = 3$ and $y = 2$.

We plot these points on graph paper and join them. This line divides the xy-plane in two parts. Clearly. (0, 0) satisfies the inequality. So, the region containing the origin is represented by the given inequation as shown in Fig. This shaded region represents the solution set.



2. Given $y \le -3$. which can be rewritten as $y = -3$. Clearly, it is a line parallel to x-axis. The line $y = -3$ divides the xy-plane into two regions one below it and other above it. We find that (0, 0) does not satisfy the inequation $y \le -3$. So, the region represented by the given inequation is the region not containing the origin as show in Fig. Clearly, it is the solution set.



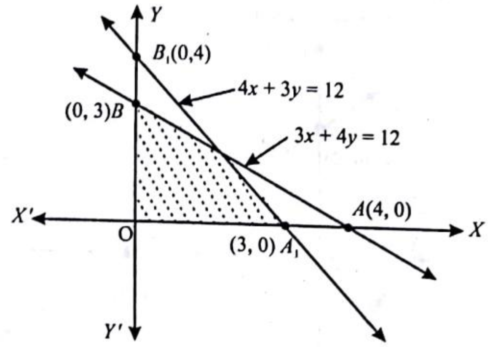
3. The inequations reduce to $3x + 4y = 12$, $4x + 3y = 12$, $x = 0$ and $y = 0$.

The line $3x + 4y = 12$ meets the coordinate axes at $A(4, 0)$ and $B(0, 3)$ Join A and B. We find that (0, 0) satisfies inequation $3x + 4y \le 12$. So, the portion containing the origin represents the solution set of the inequation $3x + 4y \le 12$.

Now, the line $4x + 3y = 12$ meets the x and y-axes $A_1(3, 0)$ and $B_1(0, 4)$ respectively. Clearly, the region containing the origin is represented by the inequation

$4x + 3y \le 12$.

Clearly, $x \ge 0$ and $y \ge 0$ represent the first quadrant. Hence, the shaded region given in fig. represents the solution set of the given linear inequations.



4. The given system of inequalities is

$x + 2 > 11$ ....(1)

$2x \le 20$ ....(2)

Now, $x + 2 > 11 \Rightarrow x > 11 - 2 \Rightarrow x > 9$

$\Rightarrow x \in (9, \infty)$

$\therefore$ Solution of inequality (1) is $x > 9$ ....(3)

and $2x \le 20 \Rightarrow x \le 10 \Rightarrow x \in (-\infty, 10]$

$\therefore$ Solution of inequality (2) is $x \le 10$ ....(4)

Clearly the common values of x satisfying (3) and (4) lie between 9 and 10.

Hence, the solution of the given system is given by $9 < x \le 10 \Rightarrow x \in (9, 10]$

5. $2x - y > 1$ ... (i) and $x - 2y < -1$ ...(ii)

Graph of inequality (i):

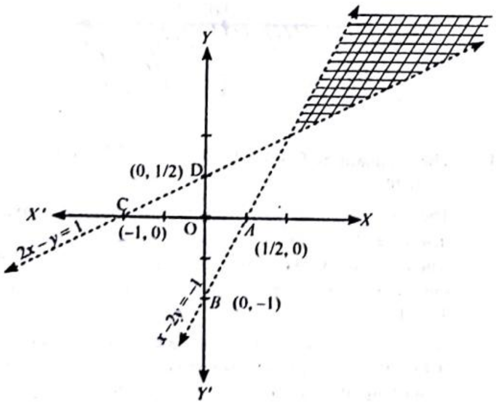
Let us draw the graph of line

$2x - y = 1$

At $y = 0$, $x = \frac{1}{2}$, the point A is $\left(\frac{1}{2}, 0\right)$

At $x = 0$, $y = -1$, the point B is $(0, -1)$

Putting $x = y = 0$ in (i), we have $0 > 1$



which is false

Hence, half plane region lying below AB is the solution region. (not including the line AB).

Graph of inequality (ii):

Let us draw the graph of the line $x - 2y = -1$

Putting $y = 0$, $x = -1$, point C is $(-1, 0)$

Putting $x = 0$, $y = \frac{1}{2}$, point D is $\left(0, \frac{1}{2}\right)$

putting $x = y = 0$ in (ii) we have $0 < -1$, which is false

Hence, half plane region lying above CD in the solution region (not including line CD). Thus the shaded region is the solution region of the given inequalities.

EXERCISE 2

MULTIPLE CHOICE QUESTIONS

1. (c) $-n \le x - 4\sqrt{n} \Rightarrow x \ge -n + 4\sqrt{n}$

$\Rightarrow x \ge 4 - (n - 4\sqrt{n} + 4)$

$\Rightarrow x \ge 4 - (\sqrt{n} - 2)^2$

Now, $(\sqrt{n} - 2)^2$ is positive since it is a perfect square, the least value of $(\sqrt{n} - 2)^2 = 0$. Thus, $x \ge 4$.

2. (d) Given, $\frac{x+3}{x-3} < 1$

Subtract 1 on both side, $\frac{x+3}{x-3} - 1 < 1 - 1$

$\Rightarrow \frac{x+3-x+3}{x-3} < 0$

$\Rightarrow \frac{6}{x-3} < 0$

$\Rightarrow$ x cannot be equal to 4.

3. (d) Given, $0 < \frac{2x-5}{2} < 7$

Multiply by 2 on each side, $0 < 2x - 5 < 14$

Add 5 on each side,

$0 + 5 < 2x - 5 + 5 < 14 + 5$

$\Rightarrow 5 < 2x < 19$

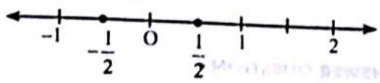
$\Rightarrow \frac{5}{2} < x < \frac{19}{2}$

So, Required sum $= \frac{5}{2} + \frac{19}{2} = \frac{24}{2} = 12$

4. (d) Consider $5 \le 2x + 7 \le 8$

$\Rightarrow -2 \le 2x \le 1$

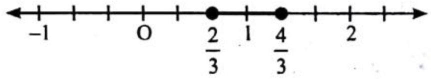
$\Rightarrow -1 \le x \le \frac{1}{2}$



Consider $7 \le 3x + 5 \le 9$

$\Rightarrow\ 2 \le 3x \le 4$

$\Rightarrow\ \frac{2}{3} \le x \le \frac{4}{3}$



There is no common solution set.

5. (a) $(x+5)-7(x-2) \ge 4x+9$

$\Rightarrow\ 19-6x \ge 4x+9$

$\Rightarrow\ 10 \ge 10x$

$\Rightarrow\ x \le 1$ .....(1)

$2(x-3)-7(x+5) \le 3x-9$

$\Rightarrow\ -5x-41 \le 3x-9$

$\Rightarrow\ -8x-41 \le -9$

$\Rightarrow\ x \ge -4$ ....(2)

Thus, from (1) and (2)

$-4 \le x \le 1$

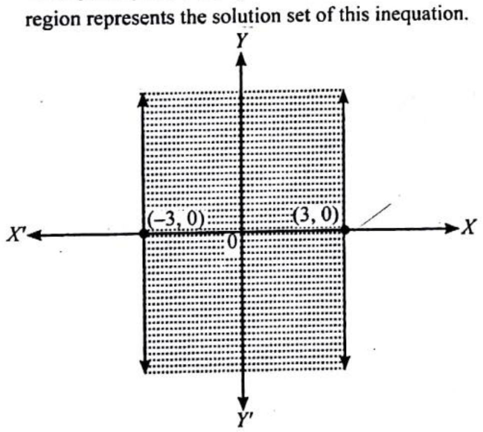
6. (a) $x > y$

$\Rightarrow -x < -y$

HOTS SUBJECTIVE QUESTIONS

1. (i) Given inequation can be rewritten as, $x = 3$. This equation represents a line parallel to y-axis at a distance of 3 units from it. The line given by $x = 3$ divides the xy-plane into two regions.

Clearly, the point $O(0, 0)$ satisfies $x \le 3$. The shaded region represents the solution set of this inequation.



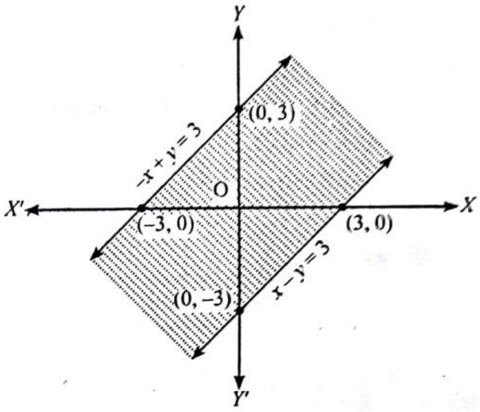
(ii) Given $|y-x| \le 3$. This inequation is equivalent to

$-3 \le y-x \le 3$ $[\because |x| \le a \Leftrightarrow -a \le x \le a]$

$\Leftrightarrow\ -3 \le y-x$ and $y-x \le 3$

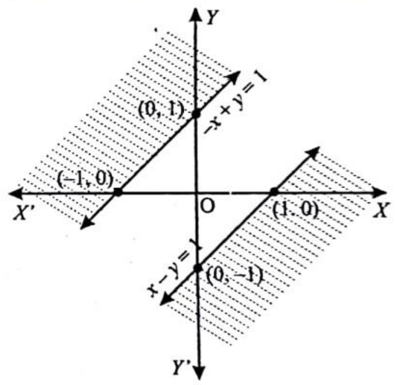
$\Leftrightarrow\ x-y-3 \le 0$ and $x-y+3 \ge 0$

Required region is the region common to the regions represented by $x-y-3 \le 0$ and $x-y+3 \ge 0$ as shown in Fig. This shaded region represents the solution set of the given inequation.



(iii) Given, $|x-y| \ge 1 \Leftrightarrow x-y \ge 1$ or $x-y \le -1$ (A)

The required region is the union of regions represented by equation (A) as shown in Fig. The shaded region represents the solution set of the given inequation.



2. $x+y \le 4,\ y \le 3,\ x \le 3,\ x+5y \ge 4,\ 6x+2y \ge 8,\ x \ge 0,\ y \ge 0$

3. We have,

Profit = Revenue – Cost

$\therefore$ To earn some profit, we must have

Revenue > Cost

$\Rightarrow\ 2x > 300 + \frac{3}{2}x$

$\Rightarrow\ 2x - \frac{3}{2}x > 300$

$\Rightarrow\ \frac{x}{2} > 300 \Rightarrow x > 600$

Hence, more than 600 cassettes must be sold.

4. Consider $\left|\dfrac{2}{x-4}\right| > 1, x \neq 4$

$\Rightarrow \dfrac{2}{|x-4|} > 1$

$\Rightarrow 2 > |x-4| \quad (\because |x-4| > 0 \text{ for all } x \neq 4)$

$\Rightarrow 4-2 < x < 4+2$

$(\because |x-a| < r \Leftrightarrow a-r < x < a+r)$

$\Rightarrow 2 < x < 6 \Rightarrow x \in (2, 6)$

But $x \neq 4$

Solution set is $(2, 4) \cup (4, 6)$

5. Given

$\dfrac{|x|-1}{|x|-2} \geq 0,$

Let $y = |x|$

$\Rightarrow \dfrac{y-1}{y-2} \geq 0$

$\Rightarrow y \leq 1$ or $y > 2 \Rightarrow |x| \leq 1$ or $|x| > 2$

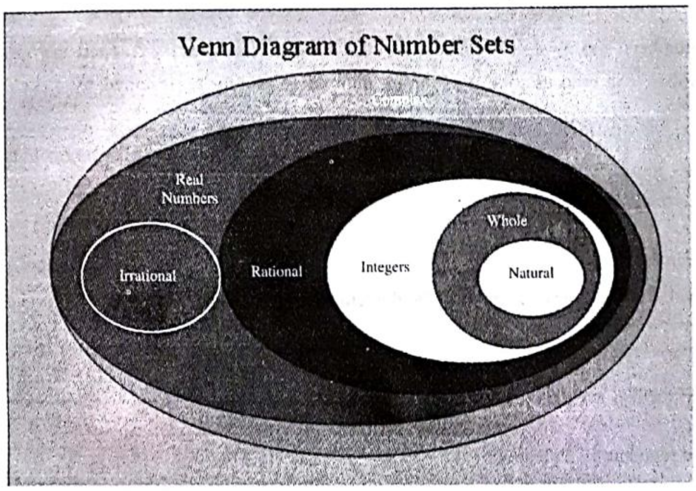
$\Rightarrow (-1 \leq x \leq 1)$ or $(x < -2$ or $x > 2)$

$\Rightarrow x \in [-1, 1]$ or $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

Hence, solution set =

$[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$



13

CHAPTER

Complex Numbers

INTRODUCTION

In this chapter, we extend the number system from real numbers to complex numbers, which have significant applications in mathematics, physics and engineering. For example, complex numbers are used to describe voltage and other electrical quantities in physics.

To define a complex number, we first introduce a new set of numbers called imaginary numbers, in which square roots of negative numbers are defined.

The basic unit of pure imaginary numbers is $\sqrt{-1}$ and it is denoted by 'i'. Thus by definition $i = \sqrt{-1}$ and $(i)^2 = -1$; $(i)^4 = 1$

If a is a positive real number, then $\sqrt{-a} = i\sqrt{a}$

In this definition, we choose to write 'i' in front of any radicals, so that expressions such as $\sqrt{a}\,i$ is not confused with $\sqrt{ai}$.

COMPLEX NUMBER

A number of the form $a + ib$, where a, b are real numbers, and $i = \sqrt{-1}$ is called a complex number, a is called the real part and b is called the imaginary part of the complex number. i is read as iota. A complex number is usually represented by z. i.e., $z = a + ib$

INTEGRAL POWERS OF IOTA

We have $i = \sqrt{-1}$ and $i^2 = -1$. So $i^3 = i^2 . i = (-1)i = -i$

and $i^4 = (i^2)^2 = (-1)^2 = 1$.

Note that i^0 is defined as 1.

To find the values of i^n, $n > 4$, we first divide n by 4. Let m be the quotient and r be the remainder.

Then $n = 4m + r$, where $0 \le r \le 3$.

$\therefore \quad i^n = i^{4m+r} = (i^4)^m i^r = (1)^m i^r = i^r \qquad [\because \; i^4 = 1]$

Thus if $n > 4$, then $i^n = i^r$, where r is the remainder when n is divided by 4.

For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

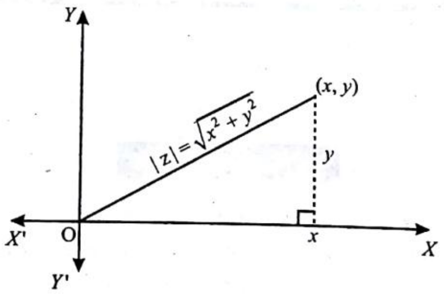
The values of the negative integral powers of i are found as given below :

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i, \qquad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i, \qquad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

MODULUS OF A COMPLEX NUMBER $z = x + iy$

$|z|$ = Distance of point (x, y) from origin in XOY-plane called Argand plane



MULTIPLICATIVE INVERSE OF A COMPLEX NUMBER

For any non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$, there exists the complex number $\frac{1}{z}$ or $z^{-1} = \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}$ called the multiplicative inverse of z such that $z \cdot \frac{1}{z} = (a + ib)\left(\frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}\right) = 1 + i0 = 1$

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBER

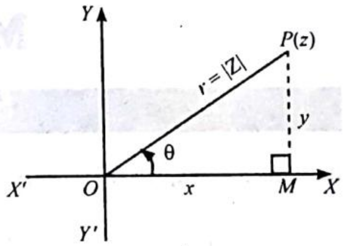
(a) Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$

(b) Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$

(c) Multiplication : $(a + ib)\;(c + id) = (ac - bd) + i(ad + bc)$

(d) Division : $\frac{a + ib}{c + id}$ (when at least one of c and d is non-zero) $= \frac{(ac + bd)}{c^2 + d^2} + \frac{i(bc - ad)}{c^2 + d^2}$

POLAR FORM OF A COMPLEX NUMBER



Every complex number $x + iy$ can be represented geometrically as a unique point $P(x, y)$ in the XOY plane whose x-coordinate representing its real part and y-coordinate representing its imaginary part.

Let the directed line segment OP be of length r and makes an angle θ with the positive direction of the x-axis (θ in radians) in anticlockwise direction

The point P can also be uniquely determined by the ordered pair of real numbers (r, θ), called the polar coordinates of the point P.

Clearly, $x = r\cos\theta,\ y = r\sin\theta$,

$$r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x}$$

Thus $z = r(\cos\theta + i\sin\theta)$ is the polar form of z. r is the modulus of the complex number z and θ is called the ARGUMENT (or AMPLITUDE) of the complex number z, denoted by arg (z) or amp (z). Hence,

$$r = |z| = \sqrt{x^2 + y^2} \text{ and } \theta = \arg(z) = \tan^{-1}\frac{y}{x}$$

$z = r(\cos\theta + i\sin\theta)$ is also written as r cis (θ).

The unique value of θ, $-\pi < \theta \le \pi$ is called the principal value of the Argument.

Note: S.I unit of angle is radian. π radian = 180°

WORKING RULE FOR FINDING PRINCIPAL ARGUMENT

Let $\tan\alpha = \left|\frac{y}{x}\right|$; where α is an +ve acute angle

(i) If $x > 0, y > 0$; then $\theta = \arg(z) = \alpha$

(ii) If $x < 0, y > 0$; then $\theta = \arg(z) = \pi - \alpha$

(iii) If $x < 0, y < 0$; then $\theta = \arg(z) = \alpha - \pi$

(iv) If $x > 0, y < 0$; then $\theta = \arg(z) = -\alpha$

(v) If $y = 0$ and $x > 0$, then $\theta = \arg(z) = 0$

(vi) If $y = 0$ and $x < 0$, then $\theta = \arg(z) = \pi$

(vii) If $x = 0$ and $y > 0$, then $\theta = \arg(z) = \pi/2$

(viii) If $x = 0$ and $y < 0$ then $\theta = \arg(z) = -\pi/2$

CONJUGATE OF A COMPLEX NUMBER

If $z = a + ib$ be a complex no. Then $\overline{z} = a - ib$ is called the conjugate of z.

Properties of conjugate of complex number:

(i) $\overline{(|\overline{z}|)} = z$

(ii) $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$

(iii) $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$

(iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$ if $z_2 \neq 0$

MISCELLANEOUS

Solved Examples

Illustration 1 : Express the complex number $\left(\frac{1}{3}+3i\right)^3$ in the form $a+ib$.

SOLUTION : $\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+3\left(\frac{1}{3}\right)^2(3i)+3.\left(\frac{1}{3}\right)(3i)^2+(3i)^3$

$$=\frac{1}{27}+i+9(-1)+27i^3=\frac{1}{27}+i-9+27i.(i^2)$$

$$=\frac{1}{27}+i-9+27i.(-1)=\frac{1}{27}+i-9-27i=-\frac{242}{27}-26i$$

It is in the form of $a+ib$, on comparing,

so, that $a=-\frac{242}{27}$, $b=-26$

Illustration 2 : Find the multiplicative inverse of $(\sqrt{5}+3i)$

SOLUTION : We have multiplicative inverse of $\sqrt{5}+3i$

$=\frac{1}{\sqrt{5}+3i}=\frac{1}{\sqrt{5}+3i}\times\frac{\sqrt{5}-3i}{\sqrt{5}-3i}$ (multiply both numerator and denominator by conjugate of $\sqrt{5}+3i$)

$$=\frac{\sqrt{5}-3i}{5-9i^2}=\frac{\sqrt{5}-3i}{5+9}=\frac{\sqrt{5}-3i}{14}=\frac{\sqrt{5}}{14}-\frac{3}{14}i$$

Illustration 3 : Express the following expression in the form of $a+ib$ $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})}$

SOLUTION : We have $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+i\sqrt{2})-(\sqrt{3}-i\sqrt{2})}=\frac{(3)^2-(i\sqrt{5})^2}{\sqrt{3}+i\sqrt{2}-\sqrt{3}+i\sqrt{2}}=\frac{9-5i^2}{2\sqrt{2}i}$

$$=\frac{9-5(-1)}{2\sqrt{2}i}=\frac{14}{2\sqrt{2}i}=\frac{7}{\sqrt{2}i}\times\frac{\sqrt{2}i}{\sqrt{2}i}=\frac{7\sqrt{2}i}{2i^2}=-\frac{7\sqrt{2}i}{2}$$

Illustration 4 : Find the modulus and the arguments of the complex numbers $z=-1+i\sqrt{3}$

SOLUTION : $z=-1+i\sqrt{3}\Rightarrow|z|=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2$

we know that the polar form of $z=r(\cos\theta+i\sin\theta)$

$\therefore$ Let $-1+i\sqrt{3}=r(\cos\theta+i\sin\theta)\Rightarrow r\cos\theta=-1$ and $r\sin\theta=\sqrt{3}$

By squaring and adding, we get

$r^2(\cos^2\theta+\sin^2\theta)=1+3 \Rightarrow r^2=4$ or $r=2$

By dividing $\frac{r\sin\theta}{r\cos\theta}=\frac{\sqrt{3}}{-1}=-\sqrt{3} \Rightarrow \tan\theta=-\tan\frac{\pi}{3}=\tan\left(\pi-\frac{\pi}{3}\right)=\tan\left(\frac{2\pi}{3}\right)$

or, $\theta=\frac{2\pi}{3}$ i.e. θ lies in 2nd quadrant.

Hence, $\arg z=\frac{2\pi}{3}$ and $|z|=2$

Illustration 5 : Convert $(1 - i)$ in the polar form

SOLUTION : We have $1 - i = r(\cos\theta + i\sin\theta) \Rightarrow r\cos\theta = 1, r\sin\theta = -1$

By squaring and adding, we get $r^2(\cos^2\theta + \sin^2\theta) = 1^2 + (-1)^2 \Rightarrow r^2 . 1 = 1 + 1 \Rightarrow r^2 = 2$

$\therefore \quad r = \sqrt{2}$. By dividing $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{-1}{1} = -1$

$\tan\theta = -1$, θ lies in fourth quadrant.

$\Rightarrow \quad \theta = -\dfrac{\pi}{4}$

$\therefore \quad$ Polar form of $1 - i = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$

Illustration 6 : Express the complex number $\left(-2-\dfrac{1}{3}i\right)^3$ in the form $a + ib$

SOLUTION : $\left(-2-\dfrac{1}{3}i\right)^3 = (-2)^3 + 3(-2)^2.\left(-\dfrac{1}{3}i\right) + 3(-2)\left(-\dfrac{1}{3}i\right)^2 + \left(-\dfrac{1}{3}i\right)^3$

$$= -8 - 4i - 6\times\frac{1}{9}(i^2) - \frac{1}{27}i^3 = -8 - 4i - \frac{2}{3}(-1) - \frac{1}{27}i.i^2$$

$$= -8 - 4i + \frac{2}{3} - \frac{1}{27}i.(-1) = -8 - 4i + \frac{2}{3} + \frac{1}{27}i = -\frac{22}{3} - \frac{107}{27}i$$

It is in the form of $a + ib$, where $a = -\dfrac{22}{3}$, $b = -\dfrac{107}{27}$

Illustration 7 : Find the multiplicative inverse of $4 - 3i$

SOLUTION : We have multiplicative inverse of $4 - 3i$

$$= \frac{1}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i}{4^2 - 9i^2} = \frac{4+3i}{16+9} = \frac{4+3i}{25} = \frac{4}{25} + i\frac{3}{25}$$

Illustration 8 : Find the multiplicative inverse of $-i$.

SOLUTION : We have multiplicative inverse of $-i = \dfrac{1}{-i}$

$$= \frac{1}{-i} \times \frac{i}{i} = \frac{-i}{i^2} = \frac{-i}{-1} = i = 0 + i . 1$$

Illustration 9 : Find the modulus and the arguments of $z = -\sqrt{3} + i$

SOLUTION : We know that the polar form $z = r(\cos\theta + i\sin\theta)$

$\therefore \quad -\sqrt{3} + i = r(\cos\theta + i\sin\theta) \Rightarrow r\cos\theta = -\sqrt{3}$ and $r\sin\theta = 1$

By squaring and adding, we get $r^2(\cos^2\theta + \sin^2\theta) = (\sqrt{3})^2 + 1$

$r^2 . 1 = 4, \quad \therefore \; r = 2$

By dividing, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{-1}{\sqrt{3}}$, θ lies in second quadrant

$\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

$\therefore \quad |z| = 2$ and $\arg z = \dfrac{5\pi}{6}$

Illustration 10 : Convert $(-1 + i)$ in the polar form

SOLUTION : We have $-1 + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r\cos\theta = -1$ and $r\sin\theta = 1$

By squaring and adding, we get $r^2(\cos^2\theta + \sin^2\theta) = (-1)^2 + 1^2 \Rightarrow r^2 .\ 1 = 1 + 1$

$\therefore \quad r^2 = 2 \qquad \therefore \quad r = \sqrt{2}$

By dividing, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{-1} = -1$

θ lies in second quadrant ; $\theta = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$\therefore$ Polar form of $(-1 + i) = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

EXERCISE 1

FIB Fill in the Blanks

DIRECTIONS: *Complete the following statements with an appropriate word/term to be filled in the blank space(s).*

1. Value of i^{135} is ______________.
2. Value of i^{19} is ___________.
3. Value of $i^n + i^{n+1} + i^{n+2} + i^{n+3} =$ _______ for all $n \in N$.
4. Value of $(-\sqrt{-1})^{4n+3} =$ ___________, $n \in N$
5. Multiplicative inverse of $3 - 2i$ is ___________.

T/F True/False

DIRECTIONS: *Read the following statements and write your answer as true or false.*

1. If z is a complex number then $\overline{(\overline{z})} = z$
2. Conjugate of $4 - 5i$ is $5 - 4i$.
3. The modulus of a complex number $z = a + ib$ is denoted by $|z|$.
4. Modulus of -4 is 4
5. If z_1 and z_2 are two complex numbers then $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$

MTC Match the Columns

DIRECTIONS: *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D, E) in Column I have to be matched with statements (p, q, r, s, t) in column II.*

1.

Column I	Column II
(A) Multiplicative inverse of $(2+\sqrt{3}i)^2$ is	(p) $5 + i$
(B) Conjugate of $\dfrac{1}{3+4i}$ is	(q) $-6 - 2i$
(C) Value of $(3 + 2i)(2 - 3i)$ is	(r) $12 - 5i$
(D) If $z_1 = -2 + 3i$ and $z_2 = 4 + 5i$ then the value of $z_1 - z_2$ is	(s) $\dfrac{1}{49} - \dfrac{4\sqrt{3}}{49}i$
(E) If $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$ then the value of $z_1 + z_2$ is	(t) $\dfrac{3}{25} + \dfrac{4}{25}i$

SAQ Short Answer Questions

DIRECTIONS: *Give answer in 2-3 sentences.*

1. Multiply $(3 - 2i)$ by its conjugate.
2. Find the least positive value of n, if $\left(\dfrac{1+i}{1-i}\right)^n = 1$
3. Find non-zero integral solutions of $|1-i|^x = 2^x$
4. Find the modulus and principal argument of $-2i$.
5. If $z_1 = 2 - i$, $z_2 = 1 + i$ find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$.
6. Find the modulus of $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$.
7. Find the conjugate of $\dfrac{1}{3+4i}$.
8. Find the multiplicative inverse of $z = 3 - 2i$.

LAQ Long Answer Questions

DIRECTIONS: *Give answer in four to five sentences.*

1. Find the modulus and principal argument of $(1 + i)$ and hence express it in the polar form.
2. If $a + ib = \dfrac{c+i}{c-i}$, where c is real, prove that : $a^2 + b^2 = 1$ and $\dfrac{b}{a} = \dfrac{2c}{c^2 - 1}$
3. Find the value of $x^3 + 7x^2 - x + 16$, when $x = 1 + 2i$
4. Evaluate : $\left[i^{18} + \left(\dfrac{1}{i}\right)^{25}\right]^3$
5. Find the modulus and argument of the complex number $\dfrac{1+2i}{1-3i}$

EXERCISE 2

MCQ Multiple Choice Questions

DIRECTIONS: *This section contains multiple choice questions. Each questions has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

1. $\left(\frac{2i}{1+i}\right)^2$

(a) i (b) $2i$

(c) $1-i$ (d) $1-2i$

2. If $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ is of the form $a+ib$, then values of a and b are

(a) $a=\frac{1}{4}, b=\frac{9}{4}$ (b) $a=-\frac{1}{4}, b=\frac{9}{4}$

(c) $a=-\frac{1}{4}, b=\frac{-9}{4}$ (d) $a=\frac{1}{4}, b=\frac{-9}{4}$

3. If $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$ then

(a) $a=2, b=-1$ (b) $a=1, b=0$

(c) $a=0, b=1$ (d) $a=-1, b=2$

4. If $z=\frac{1+7i}{(2-i)^2}$, then the polar form of z is

(a) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (b) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

(c) $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$ (d) None of these

5. $i^{57}+\frac{1}{i^{25}}$, when simplified has the value

(a) 0 (b) $2i$

(c) $-2i$ (d) 2

6. $1+i^2+i^4+i^6+\dots+i^{2n}$ is

(a) positive (b) negative

(c) 0 (d) cannot be determined

7. If $z=x+iy$ and $\left|\frac{z-5i}{z+5i}\right|=1$ then z lies on

(a) x-axis (b) y-axis

(c) line $y=5$ (d) None of these

8. For $z=\sqrt{3}-i$, the principal value of arg(z) is

(a) $-\pi/6$ (b) $-\pi/3$

(c) $-\pi/2$ (d) $-\pi/4$

9. The multiplicative inverse of $z=3-2i$, is

(a) $\frac{1}{3}(3+2i)$ (b) $\frac{1}{13}(3+2i)$

(c) $\frac{1}{13}(3-2i)$ (d) $\frac{1}{4}(3-2i)$

10. If $(x+iy)(2-3i)=4+i$, then

(a) $x=-14/13, y=5/13$ (b) $x=5/13, y=14/13$

(c) $x=14/13, y=5/13$ (d) $x=5/13, y=-14/13$

MTQ More than One Correct

DIRECTIONS: *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which ONE or MORE may be correct.*

1. Which of the following is /are not correct?

(a) $(a+ib)+(c+id)=(a+c)+i(b+d)$

(b) $(a+ib)+(c+id)=(a+ib)-(c+id)$

(c) $i^4=1$

(d) $i^4=-1$

2. Which of the following is /are correct?

(a) $i^3=-i$ (b) $i=\sqrt{-1}$

(c) $i=-\sqrt{1}$ (d) $i^3=i$

3. Which of the following is/are not correct?

(a) conjugate of $a+ib=a-ib$

(b) conjugate of $a+ib=-a-ib$

(c) conjugate of $a+ib=-a+ib$

(d) conjugate of $a+ib=a+ib$.

4. Which of the following is/are incorrect?

(a) $\bar{\bar{z}}=z$, where z is a complex number.

(b) $\overline{z_1\pm z_2}=\bar{z}_1.\bar{z}_2$

(c) $\overline{z_1\pm z_2}=\bar{z}_1+\bar{z}_2$ (d) $\overline{z_1.z_2}=\bar{z}_1.\bar{z}_2$

5. Which of the following is/are correct?

(a) $z=r(\cos\theta+i\sin\theta)$ is the polar form of z.

(b) modulus of $z=r=|z|=\sqrt{x^2+y^2}$, where $z=x+iy$

(c) $\theta=$ argument of $z=\tan^{-1}\frac{y}{x}$

(d) None of these

A&R Assertion & Reason

DIRECTIONS: *Each of these questions contains an Assertion followed by reason. Read them carefully and answer the question on the basis of following options. You have to select the one that best describes the two statements.*

(a) If both **Assertion** and **Reason** are **correct** and Reason is the **correct explanation** of Assertion.

(b) If both **Assertion** and **Reason** are correct, but Reason is **not the correct explanation** of Assertion.

(c) If **Assertion** is **correct** but **Reason** is **incorrect**.

(d) If **Assertion** is **incorrect** but **Reason** is **correct**.

1. **Assertion :** If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal then $2 = x$ and $-y = 3$.

   Reason : Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$.

2. **Assertion :** If $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other then the value of x and y is ± 1 and -4 respectively.

   Reason : If $z = a + ib$ be a complex number then $\bar{z} = a - ib$ is called the conjugate of z.

   Also, quadratic equation whose roots are α and β is given as $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

SOLUTIONS

EXERCISE-I

FILL IN THE BLANKS

1. $i^{135} = i^3 = -i$

($\because$ 135 leaves remainder as 3 when it is divided by 4.)

2. $i^{19} = i^3 = -i$

($\because$ The remainder is 3 when 19 is divided by 4)

3. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n[1+i+i^2+i^3] = i^n[1+i-1-i]$

$= i^n(0) = 0$.

4. $\left(-\sqrt{-1}\right)^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3$

$= (-i^4)^n(-i)^3 = -i^3 = i$

5. $\frac{3}{13}+\frac{2}{13}i$

TRUE/FALSE

1. True
2. False.
Required conjugate = $4 + 5i$
3. True
4. True.
Let $z = -4 + 0i$
$|z| = \sqrt{(-4)^2+(0)^2} = 4$.
5. True

MATCH THE COLUMNS

1. (A) → (s), (B) → (t), (C) → (r), (D) → (q), (E) → (p)

(A) Let $z = (2+\sqrt{3}i)^2$

$z = 4+3i^2+4\sqrt{3}i = 1+4\sqrt{3}i$

$\therefore \quad \frac{1}{z} = \frac{1-4\sqrt{3}i}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)}$

$= \frac{1}{49} - \frac{4\sqrt{3}}{49}i$

(B) Let $z = \frac{1}{3+4i}$

$\Rightarrow z = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3}{25}-\frac{4}{25}i$

$\bar{z} = \frac{3}{25}+\frac{4}{25}i$

(C) $(3+2i)(2-3i) = 6-9i+4i-6i^2$

$= 6-5i-6(-1) = 6+6-5i$

$= 12-5i$

(D) $z_1 - z_2 = (-2+3i)-(4+5i) = -6-2i$

(E) $z_1 + z_2 = (2+3i)+(3-2i) = 5+i$

SHORT ANSWER QUESTIONS

1. Conjugate of $(3-2i) = 3+2i$

$\therefore (3-2i)(3+2i) = 9-4i^2 = 13$ ($\because i^2 = -1$)

2. $\frac{1+i}{1-i} = \frac{(1+i)^2}{1-i^2}$ (By rationalizing)

$= \frac{1+2i-1}{1+1} = i$

$\therefore \quad \left(\frac{1+i}{1-i}\right)^n = 1$

$\Rightarrow i^n = 1 \Rightarrow i^n = i^4 \Rightarrow n = 4$

3. $|1-i|^x = 2^x$

$\Rightarrow \left[\sqrt{(1)^2+(-1)^2}\right]^x = 2^x \Rightarrow 2^{x/2} = 2^x$

$\Rightarrow \frac{x}{2} = x \Rightarrow 2x - x = 0 \Rightarrow x = 0$.

Hence, the given equation has no solution.

4. Let $z = -2i = 0+(-2)i$. Then,

$|z| = \sqrt{0+(-2)^2} = 2$

Clearly, the point (0, –2) representing $z = -2i$ lies on the negative side of imaginary axis.

Therefore, principal argument of z is $\frac{-\pi}{2}$.

5. $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| = \frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+i}$

$= \frac{2-i+1+i+1}{2-i-1-i+i} = \frac{4}{1-i} = \frac{4}{1-i}\times\frac{1+i}{1+i} = \frac{4(1+i)}{1-i^2}$

$= \frac{4(1+i)}{1+1} = \frac{4(1+i)}{2} = 2(1+i)$

$\therefore \left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| = |2(1+i)|$

$= 2|(1+i)| = 2.\sqrt{(1)^2+(1)^2} = 2\sqrt{2}$

6. $\frac{1+i}{1-i} = \frac{1+i}{1-i}\times\frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+i^2+2i}{1+1} = \frac{2i}{2} = i$

$\frac{1-i}{1+i} = \frac{1-i}{1+i}\times\frac{1-i}{1-i} = \frac{(1-i)^2}{1-i^2} = \frac{1+i^2-2i}{1+1} = \frac{-2i}{2} = -i$

We have, $\frac{1+i}{1-i}-\frac{1-i}{1+i} = i-(-i) = 2i \therefore \left|\frac{1+i}{1-i}-\frac{1-i}{1+i}\right| = 2$

7. $\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{1}{25}(3-4i)$

$\Rightarrow$ conjugate of $\left(\frac{1}{3+4i}\right) = \frac{1}{25}(3+4i)$

8. $z^{-1} = \frac{3}{3^2+(-2)^2} + \frac{i(-(-2))}{3^2+(-2)^2} = \frac{3}{13} + \frac{2}{13}i$

$= \frac{1}{13}(3+2i)$

LONG ANSWER QUESTIONS

1. Let $z = 1 + i$. Then,

$|z| = \sqrt{1^2+1^2} = \sqrt{2}$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$

Since the point (1, 1) representing $z = 1 + i$ lies in first quadrant, therefore $\theta = \arg(z) = \frac{\pi}{4}$

Hence, the polar form of $z = 1 + i$ is

$z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

2. We have,

$a + ib = \frac{c+i}{c-i}$

$\Rightarrow\ a + ib = \frac{(c+i)(c+i)}{(c-i)(c+i)}$

$\Rightarrow\ a + ib = \frac{(c+i)^2}{c^2-i^2}$

$\Rightarrow\ a + ib = \frac{c^2+2ic+i^2}{c^2-i^2}$

$\Rightarrow\ a + ib = \frac{c^2-1}{c^2+1} + \frac{i2c}{c^2+1}$

$\Rightarrow\ a = \frac{c^2-1}{c^2+1}$ and $b = \frac{2c}{c^2+1}$

$\Rightarrow\ a^2 + b^2 = \left(\frac{c^2-1}{c^2+1}\right)^2 + \frac{4c^2}{(c^2+1)}$

and $\frac{b}{a} = \left(\frac{2c}{c^2+1}\right) / \left(\frac{c^2-1}{c^2+1}\right)$

$\Rightarrow\ a^2 + b^2 = \frac{(c^2+1)^2}{(c^2+1)^2} = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$

3. We have, $x = 1 + 2i$

$\Rightarrow\ x - 1 = 2i$

$\Rightarrow\ (x-1)^2 = 4i^2$

$\Rightarrow\ x^2 - 2x + 1 = -4$

$\Rightarrow\ x^2 - 2x + 5 = 0$

Now, $x^3 + 7x^2 - x + 16$

$= x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$

$= x(0) + 9(0) + 12x - 29$ [$\because x^2 - 2x + 5 = 0$]

$= 12(1 + 2i) - 29$ [$\because x = 1 + 2i$]

$= -17 + 24i$

4. $\frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

$\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = \left[i^{18} + (-i)^{25}\right]^3 = \left[(-1)^9 - i(-1)^{12}\right]^3$

$= \left[(-1)^9 - i(-1)^{12}\right]^3 = [-1-i]^3$

$= (-1)^3 + (-i)^3 + 3(-1)^2(-i) + 3(-1)(-i)^2 = -1 - i^3 - 3i + 3$

$= -1 - i(i^2) - 3i + 3$

$= -[1 + i]^3 = -[1 + i^3 + 3.\, i^2.\, 1 + 3.\, 1^2 \cdot -i]$

$= -[1 - i - 3 + 3i]$

$= -[-2 + 2i]$

$= 2 - 2i$

$\therefore\ \left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = 2 - 2i$

5. We have, $z = \frac{1+2i}{1-3i} = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1-9i^2}$

$= \frac{1-6+5i}{1+9} = \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5}{10}i = -\frac{1}{2} + \frac{1}{2}i$

Put in the polar form $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow r\cos\theta = -\frac{1}{2}$ and $r\sin\theta = \frac{1}{2}$

By squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$\Rightarrow\ r^2.1 = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} \Rightarrow r^2 = \frac{1}{2}$

$\therefore\ r = \frac{1}{\sqrt{2}}$

By dividing $\frac{r\sin\theta}{r\cos\theta} = \frac{1/2}{-1/2} = -1$

$\tan\theta = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}$

$\therefore$ θ lies in second quadrant

$\therefore\ \left|\frac{1+2i}{1-3i}\right| = \frac{1}{\sqrt{2}}$ and $\arg\left|\frac{1+2i}{1-3i}\right| = \frac{3\pi}{4}$

EXERCISE 2

MULTIPLE CHOICE QUESTIONS

1. (b) $\left(\frac{2i}{1+i}\right)^2 = \frac{4i}{1+i^2+2i} = \frac{-4}{1-1+2i} = \frac{-4}{2i}$

$= \frac{-2}{i} = 2i \quad \left(\because \frac{1}{i} = -i\right)$

2. (a) $\frac{1}{1-2i} + \frac{3}{1+i} = \frac{1+i+3-6i}{(1-2i)(1+i)}$

$= \frac{4-5i}{(1-2i)(1+i)} \times \frac{3+4i}{(2-4i)}$

$= \frac{12+16i-15i-20i^2}{(1+i-2i-2i^2)(2-4i)} = \frac{32+i}{(3-i)(2-4i)}$

$= \frac{32+i}{(6-12i-2i+4i^2)} = \frac{32+i}{2-14i}$

Now, rationalize this, we get

$a = \frac{1}{4}$ and $b = \frac{9}{4}$

3. (b) $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1-i^2} = \frac{1+i^2-2i}{2} = -i$

$\therefore \quad (-i)^{100} = (i)^{100} = (i^4)^{25} = 1$

$\Rightarrow \; 1 = a + ib$

$\Rightarrow \; a = 1, b = 0.$

4. (a) $z = \frac{1+7i}{(2-i)^2} = \frac{1+7i}{3-4i} = -1 + i$ (By rationalizing)

$\therefore \; r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

Also, $z = -1 + i \Rightarrow x = -1, y = 1$

$\therefore \; \tan\theta = \left|\frac{y}{x}\right| = \left|\frac{1}{-1}\right| = 1 \Rightarrow \theta = \frac{\pi}{4}$

Since, z lies in IInd quadrant

$\therefore \; \arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, polar form of $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

5. (a) $i^{57} + \frac{1}{i^{25}} = (i^4)^{14}.i + \frac{1}{(i^4)^6 i}$

$= i + \frac{1}{i} \quad (\because i^4 = 1)$

$= i - i \quad \left(\because \frac{1}{i} = -i\right)$

$= 0$

6. (d) Given expression $= 1 + i^2 + i^4 + ... + i^{2n} = 1 - 1 + 1 -$ $1 + ... + (-1)^n$, which cannot be determined unless n is known.

7. (a) $\left|\frac{z-5i}{z+5i}\right| = 1$

$\Rightarrow |z-5i|^2 = |z+5i|^2 \Rightarrow x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow y = 0$

8. (a) Here $x = \sqrt{3}$, $y = -1 \Rightarrow \tan\theta = \left|-\frac{1}{\sqrt{3}}\right| \Rightarrow \theta = \frac{\pi}{6}$

$\Rightarrow$ Principal value of arg. $z = -\pi/6$.

(Since z lies in the fourth quadrant)

9. (b) $z^{-1} = \frac{3}{3^2+(-2)^2} + \frac{i(-(-2))}{3^2+(-2)^2} = \frac{3}{13} + \frac{2}{13}i$

$= \frac{1}{13}(3 + 2i)$

10. (b) $x + iy = \frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{13} = \frac{5+14i}{13}$

$\therefore x = 5/13, \quad y = 14/13.$

MORE THAN ONE CORRECT

1. (b, d) 2. (a, b) 3. (b, c, d)
4. (a, b) 5. (a, b, c)

ASSERTION AND REASON

1. (a) Both statements are correct.

Statement given in Reason is the correct explanation for Assertion.

Let $z_1 = 2 - iy$ and $z_2 = x + 3i$

Now, $z_1 = z_2 \Rightarrow 2 - iy = x + 3i \Rightarrow 2 = x$ and $-y = 3$

2. (a) Given,

$-3 + ix^2y = x^2 + y + 4i$

$\Rightarrow \quad -3 - ix^2y = x^2 + y + 4i$

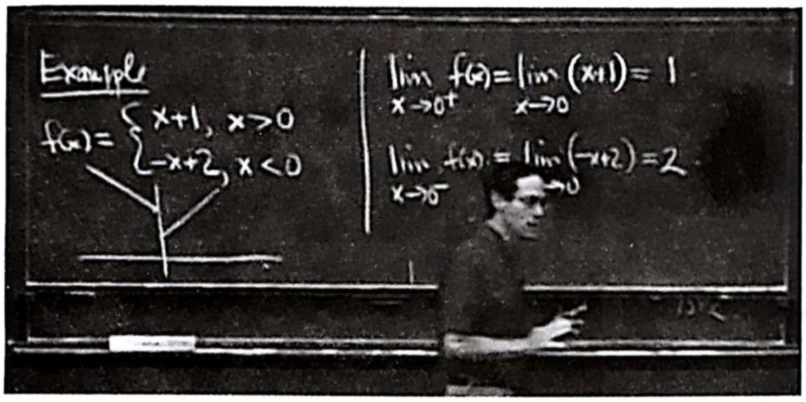
$\Rightarrow \quad -3 = x^2 + y$ and $x^2y = -4$

Now, the quadratic equation whose roots are x^2 and y is

$t^2 - (x^2 + y)t + x^2y = 0$

$\Rightarrow \quad t = -4, 1$

Since, $x^2 \geq 0 \; \therefore \; x^2 = 1, y = -4$



14

CHAPTER

Limits and Continuity

INTRODUCTION

Consider a function, $f(x) = 2x + 3$

x approaches 4 from left → 4 ← x approaches 4 from right

x	2	3.6	3.9	3.99	3.999		4.001	4.01	4.1	4.8	5
$f(x)$	7	10.2	10.8	10.98	10.998		11.002	11.02	11.2	12.6	13

$f(x)$ approaches 11 from left → 11 ← $f(x)$ approaches 11 from right

In the above table we see that when the value of x approaches 4 (but not equal to 4) from left, the value of $f(x)$ approaches 11 and when the value **of x approaches 4 (but not** *equal to 4) from right, the value of $f(x)$ even then approaches to 11.*

Here, 11 is said to be the limit of ***f(x)*** *as x approaches 4 (but not equal to 4) from either side. We can abbreviate this statement as* $\lim\limits_{x \to 4} f(x) = 11$ or $\lim\limits_{x \to 4} (2x+3) = 11$. *Thus in general, if $y = f(x)$ be a function of x and the value of the function tends to a definite unique value 'l' as x tends to a particula value 'a' (either from left or right) then the value 'l' is called the limit of $f(x)$ at $x = a$ and we write it as* $\lim\limits_{x \to a} f(x) = l$.

The word 'continuous' means without any break, gap or jump. If the graph of a function has no break, hole, gap or jump, then the function is said to be continuous.

LIMITS

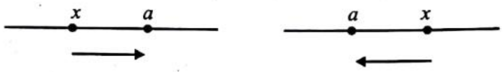
MEANING OF '$X \to a$'

Let x be a variable and a be a constant. If x assumes values nearer and nearer to 'a' then we can say 'x tends to a' and we write '$x \to a$'.

It should be noted that $x \to a$ means :

(i) x assumes values nearer and nearer to 'a' and we are not specify any manner in which x should approach to a. x may approaches to 'a' from left or right as shown in the figure.

(ii) $x \neq a$



x approaches 'a' from left x approaches 'a' from right

EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS (i.e. ONE SIDED LIMIT)

The statement $x \to a^-$ means that x is tending to 'a' from the left hand side, i.e. x is a number less than 'a' but very close to a. Therefore $x \to a^-$ is equivalent to $x = a - h$ where $h > 0$ such that $h \to 0$. Similarly $x \to a^+$ is equivalent to $x = a + h$ where $h > 0$ such that $h \to 0$. Thus, we have the following algorithms for finding left hand and right hand limits of a function at $x = a$.

ALGORITHM FOR FINDING LEFT HAND LIMIT (LHL)

To evaluate LHL of $f(x)$ at $x = a$, i.e. $\lim\limits_{x \to a^-} f(x)$ we proceed as follows :

(i) Write $\lim\limits_{x \to a^-} f(x)$

(ii) Put $x = a - h$ in $f(x)$ and replace $x \to a^-$ by $h \to 0$ to obtain $\lim\limits_{h \to 0} f(a-h)$.

(iii) Simplify $\lim\limits_{h \to 0} f(a-h)$ by using the formula for the given function.

(iv) The value obtain in step (iii) is the LHL of $f(x)$ at $x = a$.

ALGORITHM FOR FINDING RIGHT HAND LIMIT (RHL)

To evaluate RHL of $f(x)$ at $x = a$, i.e. $\lim\limits_{x \to a^+} f(x)$ we proceed as follows :

(i) Write $\lim\limits_{x \to a^+} f(x)$

(ii) Put $x = a + h$ in $f(x)$ and replace $x \to a^+$ by $h \to 0$ to obtain $\lim\limits_{h \to 0} f(a+h)$

(iii) Simplify $\lim\limits_{h \to 0} f(a+h)$ by using the formula for the given function.

(iv) The value obtained in step (iii) is the RHL of $f(x)$ at $x = a$.

EXISTENCE OF LIMIT (i.e. LIMIT FROM BOTH SIDES) OF A FUNCTION AT A POINT

The limit of a function at some point exists only when its left-hand limit & right hand limit at that point exist and are equal. If the limit of a function exists then limit of the function is equal to the left/right hand limit. Thus $\lim\limits_{x \to a} f(x)$ exists and equal if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$$

where l is called the limit of the function.

THE ALGEBRA OF LIMITS

(i) $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) . \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, provided that $\lim\limits_{x \to a} g(x) \neq 0$

(iv) $\lim\limits_{x \to a} kf(x) = k. \lim\limits_{x \to a} f(x)$, where k is a constant

(v) $\lim\limits_{x \to a} [f(x) + k] = \lim\limits_{x \to a} f(x) + k$, where k is a constant

(vi) $\lim\limits_{x \to a} |f(x)| = |\lim\limits_{x \to a} f(x)|$

(vii) $\lim\limits_{x \to a} (f(x))^{g(x)} = \left(\lim\limits_{x \to a} f(x)\right)^{\lim\limits_{x \to a} g(x)}$

SOME STANDARD LIMITS

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \sin x = 0$

(iii) $\lim_{x \to 0} \cos x = 1$

(iv) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1$

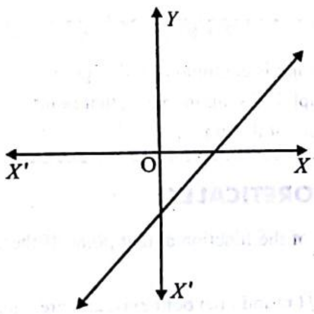
(v) $\lim_{x \to 0} \cos x = \lim_{x \to 0} \left(\frac{1}{\cos x}\right) = 1$

(vi) $\lim_{x \to 0} \tan x = 0$

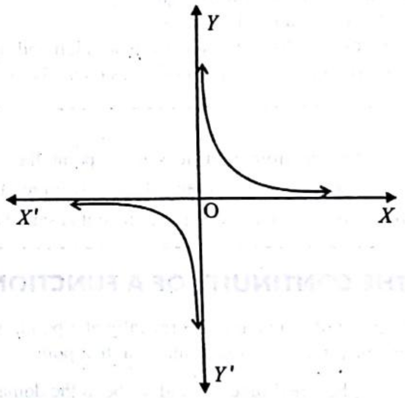
(vii) $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{x}{\tan x} = 1$

CONTINUITY

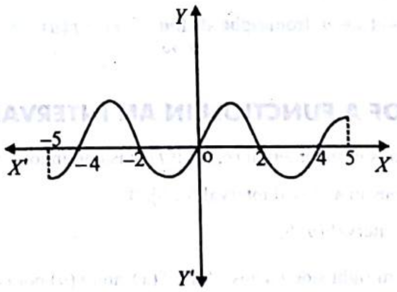
The word 'continuous' means without any break or gap. If the graph of a function has no break, hole, gap or jump, then the function is said to be continuous. A function which is not continuous is called a discontinuous function.



The graph is continuous for all values of x.



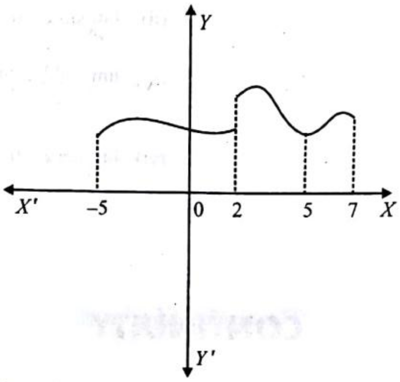
The graph is continuous for all values of x except $x = 0$.



The graph is discontinuous for all values of x less then -5 and greater than 5.

The graph is continuous for all values of $x \in (-5, 5)$ At $x = -5$, the graph is continuous from right only.

At $x = 5$, the graph is continuous from left only.



The graph is continuous for all values of $x \in (-5, 7)$

At $x = -5$, the graph is continuous from right only.
At $x = 7$, the graph is continuous from left only.
At $x = 2$ and 5, the graph is discontinuous from both left and right side.
The graph is discontinuous for all values of x less than -5 or greater than 7.

Note :
(i) If a graph is continuous from both sides at any point, then it is simply said that the graph is continuous at that point.
(ii) If a graph is discontinuous from both sides at any point, it is simply said that the graph is discontinuous at that point.
(iii) If a function is not continuous at a point, then it is said that the graph is discontinuous at that point.

TO CHECK THE CONTINUITY OF A FUNCTION AT A POINT THEORETICALLY

To check the continuity of a function theoretically at a point, we find its limit and value of the function at that point. If these two exist and are equal, then function is continuous at that point.

In other words, if $f(x)$ be a real function and 'a' be in the domain of $f(x)$ such that $\lim\limits_{x\to a} f(x)$ and $f(a)$ both exist and are equal i.e. $\lim\limits_{x\to 0} f(x) = f(a)$ then $f(x)$ is continuous at $x = a$.

The function $f(x)$ is said to be continuous at $x = a$ from left if $\lim\limits_{x\to a^-} f(x) = f(a)$

The function $f(x)$ is said to be continuous at $x = a$ from right, if $\lim\limits_{x\to a^+} f(x) = f(a)$

TO CHECK THE CONTINUITY OF A FUNCTION IN AN INTERVAL THEORETICALLY

(a) A function $f(x)$ is said to be continuous in the interval (a, b), if $f(x)$ is continuous at each and every point (a, b).

(b) A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if

(i) $f(x)$ is continuous in the open interval (a, b)

(ii) $f(x)$ is continuous at $(x = a)$ from right side means $\lim\limits_{x\to a^+} f(x)$ and $f(a)$ both exist and are equal i.e. $\lim\limits_{x\to a^+} f(x) = f(a)$

(iii) $f(x)$ is continuous at $(x = b)$ from left side means $\lim\limits_{x\to b^-} f(x)$ and $f(b)$ both exist and are equal i.e. $\lim\limits_{x\to b^-} f(x) = f(b)$.

EXAMINING THE CONTINUITY OF A FUNCTION IN A GIVEN DOMAIN THEORETICALLY

Working Rule :

(i) First of all, find the doubtful points of the given function in the given domain where the function may or may not be continuous.

(ii) $x = a$ will be a doubtful point for continuity of a function $f(x)$ if at least two of the following are different.

(a) Value of $f(x)$ at $x = a$ i.e., $f(a)$

(b) Value of $f(x)$, when $x < a$ but very close to a.

(c) Value of $f(x)$, when $x > a$ but very close to a.

(iii) Examine the continuity of $f(x)$ at all doutful points

(iv) If $f(x)$ is continuous at all doutful points, then it is continuous in its domain.

SOME KNOWN EVERYWHERE CONTINUOUS FUNCTIONS

(i) Constant Function : $f(x) = k$, where k is a constant.

(ii) Identity Function: $f(x) = x$.

(iii) Polynomial Function : A function of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$; where $a_0, a_1, a_2, \ldots\ldots, a_n$ are real numbers.

(iv) sin x and cos x functions : $f(x) = \sin x$ and $g(x) = \cos x$

BASIC RESULTS ON CONTINUOUS FUNCTIONS

If f and g are two continuous functions in an interval D, then

(a) $f + g$ is continuous on D

(b) $f - g$ is continuous on D

(c) $f \cdot g$ is continuous on D

(d) αf is continuous on D, where α is a real number.

(e) $\frac{f}{g}$ is continuous on $D - \{x : g(x) = 0\}$

(f) $\frac{1}{f}$ is continuous on $D - \{x : f(x) = 0\}$

Note : The product of one continuous and one discontinuous function may or may not be continuous.

MISCELLANEOUS

Solved Examples

Example 1 : Find the value of $\lim\limits_{x\to5}\dfrac{1-\sqrt{x-4}}{x-5}$

SOLUTION : $\lim\limits_{x\to5}\dfrac{1-\sqrt{x-4}}{x-5}=\lim\limits_{x\to5}\dfrac{1-\sqrt{x-4}}{x-5}\cdot\dfrac{1+\sqrt{x-4}}{1+\sqrt{x-4}}=\lim\limits_{x\to5}\dfrac{1-x+4}{(x-5)(1+\sqrt{x-4})}=\lim\limits_{x\to5}\dfrac{-(x-5)}{(x-5)(1+\sqrt{x-4})}$

$$=\lim_{x\to5}\frac{-1}{(1+\sqrt{x-4})}=\frac{-1}{(1+\sqrt{5-4})}=\frac{-1}{2}$$

Example 2 : Find the value of the followings :

(i) $\lim\limits_{x\to3}\dfrac{x^2+11}{x^2-4}$ (ii) $\lim\limits_{x\to1}(x^2-9)^{1/3}$ (iii) $\lim\limits_{x\to4}\dfrac{x^2-16}{x^2-2x-8}$

SOLUTION :

(i) $\lim\limits_{x\to3}\dfrac{x^2+11}{x^2-4}=\dfrac{\lim\limits_{x\to3}(x^2+11)}{\lim\limits_{x\to3}(x^2-4)}=\dfrac{(3)^2+11}{(3)^2-4}=\dfrac{20}{5}=4$

(ii) $\lim\limits_{x\to1}(x^2-9)^{1/3}=\left(\lim\limits_{x\to1}(x^2-9)\right)^{1/3}=(-8)^{1/3}=-2$

(iii) $\lim\limits_{x\to4}\dfrac{x^2-16}{x^2-2x-8}=\lim\limits_{x\to4}\dfrac{(x+4)(x-4)}{(x+2)(x-4)}=\lim\limits_{x\to4}\dfrac{x+4}{x+2}$

$=\dfrac{\lim\limits_{x\to4}(x+4)}{\lim\limits_{x\to4}(x+2)}=\dfrac{4+4}{4+2}=\dfrac{4}{3}$ [limit of polynomial functions]

Example 3 : Evaluate : $\lim\limits_{x\to2}\dfrac{x^2-3x+2}{x^2+x-6}$

SOLUTION : $\lim\limits_{x\to2}\dfrac{x^2-3x+2}{x^2+x-6}=\lim\limits_{x\to2}\dfrac{(x-2)(x-1)}{(x-2)(x+3)}=\dfrac{2-1}{2+3}=\dfrac{1}{5}$

Example 4 : Evaluate : $\lim\limits_{x\to0}\dfrac{\sqrt{1+x+x^2}-1}{x}$

SOLUTION : $\lim\limits_{x\to0}\dfrac{\sqrt{1+x+x^2}-1}{x}=\lim\limits_{x\to0}\dfrac{\sqrt{1+x+x^2}-1}{x}\cdot\dfrac{\sqrt{1+x+x^2}+1}{\sqrt{1+x+x^2}+1}=\lim\limits_{x\to0}\dfrac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)}=\lim\limits_{x\to0}\dfrac{x(1+x)}{x\left(\sqrt{1+x+x^2}+1\right)}$

$$=\lim_{x\to0}\frac{1+x}{\sqrt{1+x+x^2}+1}=\frac{1}{2}$$

Example 5 : **Evaluate :** $\lim\limits_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$.

SOLUTION : $\lim\limits_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a} = \lim\limits_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{(x+2)-(a+2)}$

$= \lim\limits_{y\to b}\dfrac{y^{5/3}-b^{5/3}}{y-b}$, where $x+2=y$, $a+2=b$. and when $x\to a, y\to ab$

$=\dfrac{5}{3}b^{5/3-1}=\dfrac{5}{3}b^{2/3}=\dfrac{5}{3}(a+2)^{2/3}$.

Example 6 : **Evaluate :** $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$

SOLUTION : $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x}{x^2}$

$=2\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}.\dfrac{\sin x}{x}\right)=2\lim\limits_{x\to 0}\dfrac{\sin x}{x}.\lim\limits_{x\to 0}\dfrac{\sin x}{x}=2\,(1)(1)=2$

Example 7 : **Evaluate :** $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$

SOLUTION : $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{\cos x}-\sin x}{\sin^3 x}=\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)}{\cos x\sin^3 x}$

$=\lim\limits_{x\to 0}\dfrac{1-\cos x}{\cos x\sin^2 x}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{\cos x(1-\cos^2 x)}$

$=\lim\limits_{x\to 0}\dfrac{1-\cos x}{\cos x(1+\cos x)(1-\cos x)}=\lim\limits_{x\to 0}\dfrac{1}{\cos x(1+\cos x)}=\dfrac{1}{2}$

Example 8 : **Determine the value of the the constant k so that function $f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3\ , & \text{if } x>2\end{cases}$ is continuous.**

SOLUTION : When $x\le 2$, we have

$f(x)=kx^2$, which being a polynomial function is continuous at each $x<2$.

When $x>2$, we have

$f(x)=3$, which being a constant function is continuous at each $x>2$.

So, consider the point $x=2$.

We have,

LHL at $(x=2)=\lim\limits_{x\to 2^-}f(x)=\lim\limits_{x\to 2}kx^2=4k$, $\quad [\because f(x)=kx^2 \text{ for } x\le 2\,]$

RHL at $(x=2)=\lim\limits_{x\to 2^+}f(x)=\lim\limits_{x\to 2}3=3$ $\quad [\because f(x)=3 \text{ for } x>2\,]$

and, $f(2)=k(2)^2=4k$

As $f(x)$ is continuous in its domain. Therefore, it is also continuous at $x=2$. Consequently,

$\lim\limits_{x\to 2^-}f(x)=f(2)=\lim\limits_{x\to 2^+}f(x)\Rightarrow 4k=3\Rightarrow k=3/4$

Example 9 : **Examine whether the fucntion f given by $f(x) = x^2$ is continuous at $x = 0$**

SOLUTION : The function is defined at the given point $x = 0$ and its value is 0.

Clearly $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$

Hence, f is continuous at $x = 0$.

Example 10: **Show that the function f given by $f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is not continuous at $x = 0$.**

SOLUTION : The function is defined at $x = 0$ and its value at $x = 0$ is 1. When $x \neq 0$, the function $f(x) = x^3 + 3$.

Hence, $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^3 + 3) = 0^3 + 3 = 3$

Since the limit of f at $x = 0$ does not coincide with $f(0)$, the function is not continuous at $x = 0$.

Example 11 : **Check the points where the constant function $f(x) = k$ is continuous.**

SOLUTION : The function is defined at all real numbers and by definition, its value at any real number equals k. Let c be any real number. Then

$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$

Since $f(c) = k = \lim_{x \to c} f(x)$ for any real number c, the function f is continuous at every real number.

Example 12 : **Prove that the identity function on real numbers given by $f(x) = x$ is continuous at every real number.**

SOLUTION : The function is clearly defined at every point and $f(c) = c$ for every real number c.

Also $\lim_{x \to c} f(x) = \lim_{x \to c} x = c$

Thus, $\lim_{x \to c} f(x) = c = f(c)$

Hence, the identity function is continuous at every real number.

Example 13 : **Show that every polynomial function is continuous.**

SOLUTION : Let a polynomial function be $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ for some natural number n, $a_n \neq 0$ and $a_i \in R$. Clearly this function is defined for every real number. For a fixed real number c, we have

$\lim_{x \to c} p(x) = p(c)$

By definiton, p is continuous at c. Since c is any real number, p is continuous at every real number and hence p is a continuous function.

Example 14 : **Let f be the function defined by $f(x) = \begin{cases} \dfrac{x^2 - 1}{x^2 - 2|x-1| - 1}, & x \neq 1 \\ 1/2, & x = 1 \end{cases}$**

Check which of the following options is correct

(a) The function is continuous for all values of x

(b) The function is continuous only for $x > 1$

(c) The function is continuous at $x = 1$

(d) The function is not continuous at $x = 1$

SOLUTION : (d) For $x<1$, $f(x)=\dfrac{x^2-1}{x^2+2x-3}=\dfrac{x+1}{x+3}$ $\therefore \lim\limits_{x\to 1^-} f(x)=\dfrac{2}{4}=\dfrac{1}{2}$

For $x>1$, $f(x)=\dfrac{x^2-1}{x^2-2x+1}=\dfrac{x+1}{x-1}$ $\therefore \lim\limits_{x\to 1^+} f(x)=\dfrac{2}{0}=\infty$

$\therefore$ The function is not continuous at $x=1$.

Example 15 : If the function $f(x)$ given by $f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1\end{cases}$ is continuous at $x=1$, find the values of a and b.

SOLUTION : We have,

(LHL at $x=1$) $=\lim\limits_{x\to 1^-} f(x)$

$=\lim\limits_{x\to 1^-}(5ax-2b)=5a-2b$

(RHL at $x=1$) $=\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(3ax+b)=3a+b$ and, $f(1)=11$.

Since $f(x)$ is continuous at $x=1$.

$\therefore \lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$

$\Rightarrow$ $5a-2b=3a+b=11$ $\Rightarrow$ $5a-2b=11$ and $3a+b=11$ $\Rightarrow$ $a=3$ and $b=2$

Example 16 : If a function f is defined as $f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$

Show that f is everywhere continuous except at $x=4$.

SOLUTION : We have, $f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$

$$\Rightarrow \quad f(x)=\begin{cases}\dfrac{-(x-4)}{x-4}=-1; & x<4\\ \dfrac{x-4}{x-4}=1\ ; & x>4\\ 0 \ ; & x=4\end{cases} \qquad \left[\because\ |x-4|=\begin{cases}-(x-4), & x<4\\ x-4, & x\geq 4\end{cases}\right]$$

When $x<4$, we have $f(x)=-1$, which, being a constant function, is continuous at each point $x<4$.

Also, when $x>4$, we have $f(x)=1$, which being a constant function, is continuous at each point $x>4$.

Let us consider the point $x=4$.

We have,

$\lim\limits_{x\to 4^-} f(x)=\lim\limits_{x\to 4}-1=-1;$

$\lim\limits_{x\to 4^+} f(x)=\lim\limits_{x\to 4} 1=1,$

and, $f(4)=0$

$\therefore$ $f(x)$ is not continuous at $x=4$.

Hence, $f(x)$ is everywhere continuous, except at $x=4$.

EXERCISE 1

Fill in the Blanks

DIRECTIONS: *Complete the following statements with an appropriate word / term to be filled in the blank space(s).*

1. Limit of difference of two functions is _______ of the limits of the functions.

2. $\lim_{x \to a} \frac{f(x)}{g(x)} =$ _______

3. For any positive integer n, $\lim_{x \to a} \frac{x^n - a^n}{x - a} =$ _______

4. $\lim_{x \to 0} \frac{\sin 4x}{\sin 2x} =$ _______

5. If f and g are two continuous functions on their domain D, then $f + g$ is _______ on D.

True / False

DIRECTIONS: *Read the following statements and write your answer as true or false.*

1. Every constant function is not continuous at all points.
2. The identity function is continuous at all points.
3. Every polynomial function is continuous at all points.
4. $f(x) = \sin x$ and $f(x) = \cos x$ are not continuous at all points.
5. Limit of sum of two functions is sum of the limits of the functions.

Match the Columns

DIRECTIONS: *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D,E) in Column I have to be matched with statements (p, q, r, s,t) in column II.*

Column I	Column II
(A) Value of $\lim_{x \to 0} \frac{(1-x)^n - 1}{x}$ is	(p) 9
(B) If $\lim_{x \to -a} \frac{x^9 + a^9}{x + a} = 9$ then the value of 'a' is	(q) 9/2
(C) Value of $\lim_{x \to 0} \frac{\sin 3x}{x}$ is	(r) $-n$
(D) Value of $\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$ is	(s) ± 1
(E) Value of $\lim_{x \to 9} \frac{x^{3/2} - 27}{x - 9}$ is	(t) 3

Short Answer Questions

DIRECTIONS: *Give answer in 2-3 sentences.*

1. Evaluate the following limits :

   (i) $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$ (ii) $\lim_{x \to 3} \left(\frac{x^4 - 81}{2x^2 - 5x - 3} \right)$

   (iii) $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

2. If $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$ and $n \in N$, then find n.

3. Evaluate : $\lim_{x \to 2} \frac{\sin 5x}{2x}$

4. Evaluate : $\lim_{x \to 0} \frac{1 - \cos 2x}{x}$

5. Discuss the continuity of the function $f(x)$ given by

   $f(x) = \begin{cases} 2 - x, & x < 2 \\ 2 + x, & x \geq 2 \end{cases}$ at $x = 2$.

6. Determine the value of k for which the following function is continuous at $x = 3$.

   $$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

EXERCISE 2

Multiple Choice Questions

DIRECTIONS: *This section contains multiple choice questions. Each questions has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

1. Let a function $f: R \to R$ satisfy the equation $f(x+y) = f(x) + f(y)$ for all x, y. If the function $f(x)$ is continuous at $x = 0$, then

(a) $f(x) = 0$ for all x.

(b) $f(x)$ is continuous for all positive real x.

(c) $f(x)$ is continuous for all x.

(d) None of these.

2. The function $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & x \neq 0 \\ \dfrac{k}{2}, & x = 0 \end{cases}$

is continuous at $x = 0$, then $k =$

(a) 3 (b) 6

(c) 9 (d) 12

3. If $f(x) = \begin{cases} ax^2 + b, & 0 \le x < 1 \\ 4, & x = 1 \\ x + 3, & 1 < x \le 2 \end{cases}$

then the value of (a, b) for which $f(x)$ cannot be continuous at $x = 1$, is

(a) (2, 2) (b) (3, 1)

(c) (4, 0) (d) (5, 2)

4. The value of a for which the function

$f(x) = \begin{cases} 5x - 4, & \text{if } 0 < x \le 1 \\ 4x^2 + 3ax, & \text{if } 1 < x < 2 \end{cases}$ is continuous at every point of its domain, is

(a) $\dfrac{13}{3}$ (b) 1

(c) 0 (d) –1

More than One Correct

DIRECTIONS: *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which ONE OR MORE may be correct.*

1. Which of the following is incorrect ?

(i) If f and g are continuous then $f + g$ is not continuous.

(ii) If f and g are continuous then $f - g$ is not continuous.

(iii) If f and g are continuous then $\dfrac{f}{g}$ is not continuous.

(iv) If f is continuous then $\dfrac{1}{f}$ is also continuous provided $f \neq 0$

2. Which of the following is incorrect ?

(i) $\lim\limits_{x \to a} (f \pm g)(x) \neq \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} f(x).g(x) \neq \lim\limits_{x \to a} f(x). \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} k.f(x) = k. \lim\limits_{x \to a} f(x)$, where k is a constant

(iv) $\lim\limits_{x \to a} |f(x)| \neq \left|\lim\limits_{x \to a} f(x)\right|$

3. Which of the following is correct ?

(a) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (b) $\lim\limits_{x \to 0} \dfrac{x}{\tan x} = 1$

(c) $\lim\limits_{x \to 0} \dfrac{x}{\sin x} \neq 1$ (d) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

4. Which of the following function is/are continuous everywhere ?

(a) $f(x) = \sin x - \cos x$

(b) $f(x) = \sin x \cos x$

(c) $f(x) = c$, where c is a constant

(d) None of the above

PBQ Passage Based Questions

DIRECTIONS: *Study the given paragraph(s) and answer the following questions.*

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ attains $\frac{0}{0}$ form, then $x-a$ must be a factor of numerator and denominator which can be cancelled out.

For example.

$\lim_{x\to 2}\frac{x^2+3x-10}{x^2-5x+6}$. It is $\frac{0}{0}$ form, hence $(x-2)$ must be a factor of numerator as well as denominator

$= \lim_{x\to 2}\frac{(x-2)(x+5)}{(x-2)(x-3)} = \lim_{x\to 2}\frac{x+5}{x-3} = -7$

1. Value of $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}$ is

(a) $\frac{-1}{4}$ (b) $\frac{1}{4}$

(c) 4 (d) -4

2. Value of $\lim_{x\to 1}\frac{x^3-1}{x-1}$ is

(a) -3 (b) 3

(c) 6 (d) -6

3. Value of $\lim_{x\to\frac{1}{2}}\frac{8x^3-1}{16x^4-1}$ is

(a) $\frac{4}{3}$ (b) $\frac{1}{2}$

(c) $\frac{-4}{3}$ (d) $\frac{3}{4}$

A&R Assertion & Reason

DIRECTIONS: *Each of these questions contains an Assertion followed by reason. Read them carefully and answer the question on the basis of following options. You have to select the one that best describes the two statements.*

(a) If both **Assertion** and **Reason** are **correct** and Reason is the **correct explanation** of Assertion.

(b) If both **Assertion** and **Reason** are correct, but Reason is **not the correct explanation** of Assertion.

(c) If **Assertion** is **correct** but **Reason** is **incorrect**.

(d) If **Assertion** is **incorrect** but **Reason** is **correct**.

1. **Assertion :** $f(x) = \sin x + \cos x$ is continuous everywhere.

Reason : If $f(x)$ and $g(x)$ are two everywhere continuous then $f(x) + g(x)$ is also continuous everywhere.

2. **Assertion :** If $\lim_{x\to 1}\frac{x^4-1}{x-1} = \lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$ then the value of k is $\frac{8}{3}$.

Reason : For any positive integer n, $\lim_{x\to a}\frac{x^n-a^n}{x-a} = n.a^{n-1}$ [n can be any rational number also]

3. **Assertion :** Value of $\lim_{x\to 0}\frac{\sin 2x+\sin 3x}{2x+\sin 3x}$ is 1.

Reason : $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{x}{\sin x} = 1$

SOLUTIONS

EXERCISE – I

FILL IN THE BLANKS

1. difference 2. $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ 3. $n.a^{n-1}$

4. 2 5. Continuous

TRUE/FALSE

1. False 2. True 3. True

4. False 5. True

MATCH THE COLUMNS

Sol. (A) → r; (B) → s; (C) → t; (D) → p; (E) → q

(A) $\lim_{x\to 0}\dfrac{(1-x)^n-1}{x} = \lim_{x\to 0}\dfrac{(1-x)^n-1}{x+1-1}$

$= -\lim_{x\to 0}\dfrac{(1-x)^n-1}{(1-x)-1} = -n(1)^{n-1} = -n$

(B) $\lim_{x\to -a}\dfrac{x^9-(-a)^9}{x-(-a)} = 9 \Rightarrow 9.(-a)^{9-1} = 9$

$\Rightarrow 9.a^8 = 9 \Rightarrow a^8 = 1 \Rightarrow a = \pm 1$

(C) $\lim_{x\to 0}\dfrac{\sin 3x}{x} = \lim_{x\to 0}\dfrac{3\sin 3x}{3x}$

$= 3.\lim_{3x\to 0}\dfrac{\sin 3x}{3x} = 3.1 = 3$

(D) $\lim_{x\to 0}\dfrac{\sin^2 3x}{x^2} = \lim_{x\to 0}\dfrac{9\sin^2 3x}{9.x.x}$

$= 9\lim_{x\to 0}\dfrac{\sin^2 3x}{9x^2} = 9\times 1 = 9$

(E) $\lim_{x\to 9}\dfrac{x^{3/2}-27}{x-9} = \lim_{x\to 9}\dfrac{x^{\frac{3}{2}}-9^{\frac{3}{2}}}{x-9}$

$= \dfrac{3}{2}.(9)^{\frac{3}{2}-1} = \dfrac{3}{2}(9)^{\frac{1}{2}} = \dfrac{3}{2}\times 3 = \dfrac{9}{2}$

SHORT ANSWER QUESTIONS

1.

(i) $\lim_{x\to 0}\dfrac{(x+1)^5-1}{x} = \lim_{x\to 0}\dfrac{\left(1+5x+\frac{5.\,4}{1.\,2}x^2+.....+x^5\right)-1}{x}$

$= \lim_{x\to 0}\dfrac{5x+10x^2+.......+x^5}{x}$

$= \lim_{x\to 0}(5+10x+.....+x^4) = 5$

(ii) $\lim_{x\to 3}\left(\dfrac{x^4-81}{2x^2-5x-3}\right) = \lim_{x\to 3}\dfrac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$

$= \lim_{x\to 3}\dfrac{(x+3)(x^2+9)}{2x+1} = \dfrac{6\times 18}{7} = \dfrac{108}{7}$

(iii) $\lim_{x\to -2}\dfrac{\frac{1}{x}+\frac{1}{2}}{x+2} = \lim_{x\to -2}\dfrac{2+x}{2x}\div(x+2)$

$= \lim_{x\to -2}\dfrac{2+x}{2x}\times\dfrac{1}{(x+2)}$

$= \lim_{x\to -2}\left(\dfrac{1}{2x}\right) = -\dfrac{1}{4}$

2. $\lim_{x\to 2}\dfrac{x^n-2^n}{x-2} = 80$

$\Rightarrow n.2^{n-1} = 80 \Rightarrow n.2^n = 160.\ 2^5.5 \Rightarrow n = 5.$

3. $\lim_{x\to 2}\dfrac{\sin 5x}{2x} = \lim_{x\to 2}\left(\dfrac{5}{5}.\dfrac{\sin 5x}{2x}\right) = \dfrac{5}{2}\lim_{x\to 0}\dfrac{\sin 5x}{5x} = \dfrac{5}{2}(1) = \dfrac{5}{2}$

$\left[\because \lim_{x\to 0}\dfrac{\sin 5x}{5x} = 1\right]$

4. $\lim_{x\to 0}\dfrac{1-\cos 2x}{x} = \lim_{x\to 0}\dfrac{2\sin^2 x}{x}$

$= 2\lim_{x\to 0}\left(\dfrac{\sin x}{x}.\sin x\right) = 2\left(\lim_{x\to 0}\dfrac{\sin x}{x}\right)\left(\lim_{x\to 0}\sin x\right)$

$= 2(1)(0) = 0$

5. We have,

(LHL at $x = 2$)

$= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2-x$ [$\because f(x) = 2-x$ for $x < 2$]

$= 2-2=0$

and (RHL at $x = 2$)

$= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2+x$ [$\because f(x) = 2+x$ for $x \geq 2$]

$= 2+2=4$

$\because \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

Hence, $f(x)$ is not continuous at $x = 2$.

6. Since $f(x)$ is continuous at $x = 3$. Therefore.

$\lim_{x\to 3} f(x) = f(3)$

$\Rightarrow \lim_{x\to 3} f(x) = k$ [$\because f(3) = k$]

$\Rightarrow \lim_{x\to 3} \frac{x^2-9}{x-3} = k$

$\Rightarrow \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = k$

$\Rightarrow \lim_{x\to 3}(x+3) = k \Rightarrow 6 = k$

Thus, $f(x)$ is continuous at $x = 3$, if $k = 6$

EXERCISE – 2

MULTIPLE CHOICE QUESTIONS

1. **(c)** Since $f(x)$ is continuous at $x = 0$

$\lim_{x\to 0} f(x) = f(0)$

Take any point $x = a$, then at $x = a$

$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$

$= \lim_{h\to 0}[f(a) + f(h)]$ [$\because f(x+y) = f(x) + f(y)$]

$= f(a) + \lim_{h\to 0} f(h) = f(a) + f(0) = f(a+0) = f(a)$

$\therefore$ $f(x)$ is continuous at $x = a$. Since $x = a$ is any arbitrary point, therefore $f(x)$ is continuous for all x.

2. (b) **3.** (d) **4.** (d)

MORE THAN ONE CORRECT

1. (a,b,c) **2.** (a, b, d) **3.** (a,b,d)

4. (a,b,c)

PASSAGE BASED QUESTIONS

1. **(a)** $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x-3)}{(x+2)(x-2)}$

$= \lim_{x\to 2} \frac{x-3}{x+2} = \frac{2-3}{2+2} = \frac{-1}{4}$

2. **(b)** $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$

$= \lim_{x\to 1}(x^2+x+1) = 1^2+1+1 = 3$

3. **(d)** $\lim_{x\to \frac{1}{2}} \frac{8x^3-1}{16x^4-1} = \lim_{x\to \frac{1}{2}} \frac{(2x)^3-1^3}{(4x^2)^2-1^2}$

$= \lim_{x\to \frac{1}{2}} \frac{(2x-1)(4x^2+2x+1)}{(4x^2+1)(2x-1)(2x+1)}$

$= \lim_{x\to \frac{1}{2}} \frac{4x^2+2x+1}{(4x^2+1)(2x+1)} = \frac{3}{4}$

ASSERTION AND REASON

1. **(a)** $\sin x$ and $\cos x$ are continuous everywhere.

$\therefore$ $\sin x + \cos x$ also continuous everywhere.

2. **(a)** L.H.S $= \lim_{x\to 1} \frac{x^4-1}{x-1} = 4(1)^3 = 4$

RHS $= \lim_{x\to k} \frac{x^3-k^3}{(x-k)} \cdot \frac{(x-k)}{(x^2-k^2)} = \frac{3k^2}{2k} = \frac{3k}{2}$

LHS = RHS $\Rightarrow$ $4 = \frac{3k}{2} \Rightarrow k = \frac{8}{3}$

3. **(a)** $\lim_{x\to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x} = \lim_{x\to 0} \frac{\frac{2x\sin 2x}{2x} + \frac{3x\sin 3x}{3x}}{2x + 3x\frac{\sin 3x}{3x}}$

N^r $\lim_{x\to 0} 2x . \lim_{x\to 0} \frac{\sin 2x}{2x} + \lim_{x\to 0} 3x . \lim_{x\to 0} \frac{\sin 3x}{3x}$

$\lim_{x\to 0} 2x + \lim_{x\to 0} 3x$ $\left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$

$\lim_{x\to 0}(2x+3x) = \lim_{x\to 0} 5x$

$D^r = \lim_{x\to 0} 2x + \lim_{x\to 0} 3x . \lim_{x\to 0} \frac{\sin 3x}{3x}$

$= \lim_{x\to 0}(2x+3x) = \lim_{x\to 0} 5x$

Now, $\frac{N^r}{D^r} = \lim_{x\to 0} \frac{5x}{5x} = \lim_{x\to 0} 1 = 1$

Matrix m × n

where m is the number of rows, and n is the number of columns.

$$\begin{array}{cc} & \text{COLUMNS} \\ A = & \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix} \end{array} \text{ ROWS}$$

$A = [a_{ij}]_{m \times n}$ where i represents the row position and j represents the column position.

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CHAPTER

Matrices and Determinants

INTRODUCTION

Matrix is a powerful tool of modern mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evaluation of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Study of matrices is becoming important day by day due to its wide applications not only in mathematics, but also in physical science, engineering, genetics, economics, industrial management, sociology and modern psychology.

MATRICES

Suppose your school organizes an inter-school debate competition, where any participant can speak in either of the six languages Hindi, English, Malayalam, Telugu, Tamil or Marathi. Various schools send the list of their students who wish to participate in the debate competition. Out of all the entries received, let us consider the list of participants received from three schools A, B and C. The list received from the school A consists of 2 participants in English, 4 in Hindi, 1 in Malayalam and 1 in Telugu. In the list of school B, 5 participants are in English, 3 are in Hindi, 1 in Malayalam, 2 in Telugu and 1 in Marathi. As per the list received from the school C, 6 participants are in English, 1 in Hindi, 2 each in Malayalam and Telugu, 3 in Tamil and 4 in Marathi.

The information given in the above example, can be put in a compact way in a tabular form as follows:

Name of the School	Number of speakers (language wise)					
	English	Hindi	Malayalam	Telugu	Tamil	Marathi
A	2	4	1	1	0	0
B	5	3	1	2	0	1
C	6	1	2	2	3	4

The above arranged form of numbers is known as rectangular array. In the array, the lines from top to bottom are called columns whereas the lines from left to right are called rows. If we consider the numbers only, then 2, 4, 1, 1, 0 and 0 constitute the first row (i.e. the top row); 5, 3, 1, 2, 0 and 1 constitute the second row and so on; whereas 2, 5 and 6 are elements of the first column (i.e. extreme left column); 4, 3 and 1 are in second column and so on. Thus in the above arrangement there are 3 rows and 6 columns and therefore, total number of elements is $3 \times 6 = 18$.

Now we write the data given in the above arrangement in a capital or small bracket without any top or left heading as shown below:

$$\begin{bmatrix} 2 & 4 & 1 & 1 & 0 & 0 \\ 5 & 3 & 1 & 2 & 0 & 1 \\ 6 & 1 & 2 & 2 & 3 & 4 \end{bmatrix}$$

The above arrangement in a capital bracket is an example of a matrix.

DEFINITION OF A MATRIX

A system of any mn numbers symboles or expressions arranged in a capital or small bracket in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or simply an $m \times n$ matrix. Note that $m \times n$ is read as 'm by n'. The matrix is often denoted by capital letter English alphabets, like $A, B, C, \ldots\ldots, X, Y, Z$ etc. The entries are enclosed within square brackets or small brackets. For example the matrix shown above can be written as

$$A = \begin{bmatrix} 2 & 4 & 1 & 1 & 0 & 0 \\ 5 & 3 & 1 & 2 & 0 & 1 \\ 6 & 1 & 2 & 2 & 3 & 4 \end{bmatrix} \text{ or as } A = \begin{pmatrix} 2 & 4 & 1 & 1 & 0 & 0 \\ 5 & 3 & 1 & 2 & 0 & 1 \\ 6 & 1 & 2 & 2 & 3 & 4 \end{pmatrix}$$

In this book, we shall prefer the first notation and represent the matrices in the square brackets. The students may observe that in the matrix A as written above, there are 3 rows and 6 columns. Therefore, A is 3×6 matrix or in other words say that the order of the matrix A is 3×6.

Each number or entry in the matrix is called an element of the matrix. An element of a matrix is generally denoted by a_{ij}, where the first subscript i of a_{ij} denotes the row number and the second subscript i of a_{ij} denotes the column number. For example, a_{13} denote the element of 1st row and 3rd column. Similarly, a_{32} denotes the element of 3rd row and 2nd column. Thus in a 3×3 matrix, the elements can be symbolically written as under :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Similarly, in case of $m \times n$ matrix, the elements can be symbolically written as :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

In this matrix, there are m rows and n columns.

After having the above idea, you can also represent a matrix by using the symbols $[a_{ij}]$ or (a_{ij}). Generally, we use the notation $[a_{ij}]$ to denote a matrix. However, this notation is incomplete unless we talk about the number of rows and columns of the matrix. Therefore, to have a clear cut idea of a matrix, we write $[a_{ij}]_{m \times n}$. Where $m \times n$ is the order of the matrix which mean, there are m rows and n columns in the matrix.

EQUALITY OF TWO MATRICES

Two matrices are said to be equal if they are of same order and each element of one matrix is equal to the corresponding element of other matrix.

Let us consider the two matrices

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ 4 & 0 \\ 3 & 7 \end{bmatrix}$$

In the matrix A there are 2 rows and 3 columns, whereas in B there are 3 rows and 2 columns. Observe that the entries of 1st and 2nd rows of the matrix A are same as the entries in the 1st and 2nd column of matrix B respectively. But these two matrices A and B are not equal because their orders 2×3 and 3×2 are not equal. Let us find out the values of a, b, c, d and k. If the following matrices X and Y are equal.

$$X = \begin{bmatrix} 1 & 7 & a \\ -4 & 3 & 0 \\ b & -2 & 5 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & c & 13 \\ -4 & k & 0 \\ 12 & d & 5 \end{bmatrix}$$

Note that both of the matrices X and Y are of the same order 3×3. So, two matrices X and Y will be equal only if their corresponding entries (or elements) will be equal. Hence the two matrices X and Y to be equal, $a = 13$, $b = 12$, $c = 7$, $d = -2$ and $k = 3$.

TYPES OF MATRICES

ROW MATRIX

A matrix, in which there is only one row, is called a row matrix. For example, [3 6 2] is a row matrix.

COLUMN MATRIX

A matrix, in which there is only one column, is called a column matrix. For example, $\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$ is a column matrix.

RECTANGULAR MATRIX

A matrix, in which unequal number of rows and columns is called a rectangular matrix. For example $\begin{bmatrix} 2 & 4 & 5 \\ -1 & 8 & 7 \end{bmatrix}$ is a rectangular matrix because the number of rows is 2, which is not equal to the number of columns, i.e., 3.

SQUARE MATRIX

A matrix in which the number of rows equal to the number of columns is called a square matrix. For example,

$$\begin{bmatrix} 1 & 0 & -3 \\ 6 & 1 & 7 \\ 2 & i & 5 \end{bmatrix}, \begin{bmatrix} \frac{1}{7} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{4} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ all are square matrices.}$$

DIAGONAL MATRIX

The elements a_{ij} of any square matrix $[a_{ij}]$ are called diagonal elements if $i = j$. The line along which the elements a_{ij} (where $i = j$) lie is called the ***Principal diagonal or simply diagonal*** of the matrix.

A square matrix in which all elements except the principle diagonal elements are zero is known as diagonal matrix. For example,

$X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ is a diagonal matrix.

UNIT OR IDENTITY MATRIX (I)

A square matrix, in which all the elements in the principal diagonal are unity and all other elements are zero, is called a unit matrix.

For example, $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3×3 or simply say 3.

Note : If number of rows and columns of a matrix are equal, then we inspite of writing both number of rows and columns to write the order of the matrix as $m \times m$ we write only m.

In other words, we can say that any square matrix $[a_{ij}]$ is called a unit matrix, *f* if

$\begin{cases} 1, \text{ whenever } i = j \\ 0, \text{ whenever } i \neq j \end{cases}$

Unit matrix is also called identity matrix. Identity matrix is represented by the English capital letter 'I'.

UPPER TRIANGULAR MATRIX

A square matrix A, whose elements $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix. For example, $A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & -8 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix.

LOWER TRIANGULAR MATRIX

A square matrix A, whose elements $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix. For example, $A = \begin{bmatrix} 8 & 0 & 0 \\ 7 & 4 & 0 \\ 4 & -1 & 5 \end{bmatrix}$ is a lower triangular matrix.

Note : Note that a diagonal matrix satisfies the property of being a lower triangular matrix as well as of being an upper triangular matrix. Therefore, we can define it in other way also as :

A square matrix which is both upper and lower triangular is called a diagonal matrix.

SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is called symmetric if $a_{ij} = a_{ji}$, for all values of i and j.

In other words, we can say that a square matrix A is symmetric matrix if $A' = A$

For example, $A = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 2 & 5 \\ 0 & 5 & 7 \end{bmatrix}$ is a symmetric matrix

Note : If A is a symmetric matrix, then kA is also symmetric, where k is a scalar.

SKEW-SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if $a_{ij} = -a_{ji}$, for all values of i and j.

In other words, we can say that a square matrix A is skew-symmetric if $A' = -A$

For example, $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Note : (i) If A is a skew-symmetric matrix, then kA is also skew-symmetric, where k is a scalar.

(ii) Each diagonal element of a skew-symmetric matrix is 0.

ALGEBRA OF MATRICES

ADDITION OF MATRICES

Addition of two matrices is defined only when their ordered are same.

If A and B are two matrices of order $m \times n$, then the matrix $A + B$ is defined as the matrix, each element of which is the sum of the corresponding elements of A and B. Thus if

$$A = \begin{bmatrix} 1 & 7 & 2 \\ -4 & 5 & 9 \\ 6 & 0 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 & 2 \\ -4 & 3 & 6 \\ 5 & -2 & 1 \end{bmatrix}, \text{ then}$$

$$A + B = \begin{bmatrix} 1+1 & 7+4 & 2+2 \\ (-4)+(-4) & 5+3 & 9+6 \\ 6+5 & 0+(-2) & 8+1 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 4 \\ -8 & 8 & 15 \\ 11 & -2 & 9 \end{bmatrix}$$

From this example, it is clear that the matrix obtained after addition will be of the same order as that of the matrices which are added.

SUBTRACTION OF MATRICES

Subtraction of two matrices is defined only when their ordered are same.

If A and B are two matrices of order $m \times n$, then the matrix $A - B$ is defined as the matrix, each element of which is obtained by subtracting the element of B from the corresponding element of A. Thus if

$$A = \begin{bmatrix} 1 & 7 & 2 \\ -4 & 5 & 9 \\ 6 & 0 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 & 2 \\ -4 & 3 & 6 \\ 5 & -2 & 1 \end{bmatrix}, \text{ then}$$

$$A - B = \begin{bmatrix} 1-1 & 7-4 & 2-2 \\ (-4)-(-4) & 5-3 & 9-6 \\ 6-5 & 0-(-2) & 8-1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

From this example, it is clear that the matrix obtained after subtraction will be of the same order as that of the matrices which are subtracted.

MULTIPLICATION OF A MATRIX BY A SCALAR

If X is a matrix and k is a scalar (or say k is any number), then kX is a matrix, each element of which is k times the corresponding element of the matrix X. Thus, if

$$X = \begin{bmatrix} 1 & 2 & 5 \\ -4 & 3 & 6 \\ 5 & -2 & 1 \end{bmatrix}, \text{ then}$$

$$kX = \begin{bmatrix} k & 2k & 5k \\ -4k & 3k & 6k \\ 5k & -2k & k \end{bmatrix}$$

Note that the matrix kX is of the same order as that of the matrix X.

MULTIPLICATION OF TWO MATRICES

Before learning the procedure of multiplication of two matrices, the readers must aware of the fact that any two matrices can not be multiplied. If A and B are two matrices, then the multiplication of two matrices A and B (i.e. product AB) is possible (or defined) only when number of columns in the first matrix A is equal to the number of rows in the second matrix B.

If A is a matrix of order $l \times m$ and B is a matrix of order $m \times n$ then $A.B$ is also a matrix of order $l \times m$ [i.e. (number of rows of first matrix) × (number of columns of second matrix)]

For example, if A is a 2×2 matrix and B is a 2×3 matrix, then the number of columns 3 of matrix A is equal to the number of rows of matrix B and hence $A.B$ is defined and it will be of order 2×5. But $B.A$ is not possible because the number of columns 5 of matrix B is not equal to number of rows of matrix A.

Consider two matrices C and D of same order 2×2, $C = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & -2 \\ 7 & 6 \end{bmatrix}$. Suppose that $X = C.D$. Here A and B both matrices are of order 2×2. Therefore, the order of X will be 2×2. Note that, here it is possible to find the $D.C$ also, the matrix X will be of the following type :

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \text{ where}$$

x_{11} = (row 1 of C) (column 1 of D)

$$= [1 \;\; 3]\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$= \begin{pmatrix} \text{First elements of first} \\ \text{row of first matrix} \end{pmatrix} \times \begin{pmatrix} \text{First element of first column} \\ \text{of second matrix} \end{pmatrix} + \begin{pmatrix} \text{Second element of first} \\ \text{row of first matrix} \end{pmatrix} \times \begin{pmatrix} \text{Second element of first} \\ \text{column of second matrix} \end{pmatrix}$$

$$= 1 \times 4 + 3 \times 7 = 4 + 21 = 25$$

x_{12} = (row 1 of C) (column 2 of D)

$$= [1 \;\; 3]\begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\begin{pmatrix} \text{First elements of first} \\ \text{row of first matrix} \end{pmatrix} \times \begin{pmatrix} \text{First element of first column} \\ \text{of second matrix} \end{pmatrix} + \begin{pmatrix} \text{Second element of first} \\ \text{row of first matrix} \end{pmatrix} \times \begin{pmatrix} \text{Second element of first} \\ \text{column of second matrix} \end{pmatrix}$$

$$= 1 \times (-2) + 3 \times 6 = -2 + 18 = 16$$

x_{21} = (row 2 of A) (column 1 of B)

$$= [2 \;\; 5]\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{pmatrix} \text{First elements of first} \\ \text{row of first matrix} \end{pmatrix} \times \begin{pmatrix} \text{First element of first column} \\ \text{of first matrix} \end{pmatrix} + \begin{pmatrix} \text{Second element of first} \\ \text{row of first matrix} \end{pmatrix} \times \begin{pmatrix} \text{Second element of first} \\ \text{column of second matrix} \end{pmatrix}$$

$$= 2 \times 4 + 5 \times 7 = 8 + 35 = 43$$

x_{22} = (row 2 of C) (column 2 of D)

$= [2\ \ 5]\begin{bmatrix}-2\\6\end{bmatrix}$

$\begin{pmatrix}\text{First elements of first}\\ \text{row of first matrix}\end{pmatrix}\times\begin{pmatrix}\text{First element of first column}\\ \text{of first matrix}\end{pmatrix}+\begin{pmatrix}\text{Second element of first}\\ \text{row of first matrix}\end{pmatrix}\times\begin{pmatrix}\text{Second element of first}\\ \text{column of second matrix}\end{pmatrix}$

$= 2\times(-2)+5\times 6 = -4+30 = 26$

Hence, $X = C.D. = \begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix} = \begin{bmatrix}25 & 16\\ 43 & 26\end{bmatrix}$

Now find out the product of the matrices C and D in reverse order, i.e., $D.C.$ Suppose we denote it by Y. Use the similar procedure as explained above. We get,

$$Y = D.C. = \begin{bmatrix}0 & 2\\ 19 & 51\end{bmatrix}$$

Note here that $X \neq Y$, that is, $C \times D = D \times C$.

Hence, we conclude that multiplication of two matrices is, in general, not commutative.

Hence, even if the product of two matrices C and D exists in both order $C.D.$ and $D.C.$, even then $C.D. \neq D.E.$

Illustration 1: Find the product $A.B.$ of two matrices A and B, where

$$A = \begin{bmatrix}1 & 1\\ 2 & 3\\ 1 & 2\end{bmatrix} \text{ and } B = \begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$$

SOLUTION :

If $AB = \begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\\ x_{31} & x_{32}\end{bmatrix}$, then

$$x_{11} = [1\ \ 1]\begin{bmatrix}2\\1\end{bmatrix} = 1\times 2+1\times 1 = 3$$

$$x_{12} = [1\ \ 1]\begin{bmatrix}3\\4\end{bmatrix} = 1\times 3+1\times 4 = 7$$

$$x_{21} = [2\ \ 3]\begin{bmatrix}2\\1\end{bmatrix} = 2\times 2+3\times 1 = 7$$

$$x_{22} = [2\ \ 3]\begin{bmatrix}3\\4\end{bmatrix} = 2\times 3+3\times 4 = 18$$

$$x_{31} = [1\ \ 2]\begin{bmatrix}1\\2\end{bmatrix} = 1\times 1+2\times 2 = 5$$

$$x_{32} = [1\ \ 2]\begin{bmatrix}3\\4\end{bmatrix} = 1\times 3+2\times 4 = 11$$

$$\therefore\ AB = \begin{bmatrix}3 & 7\\ 7 & 18\\ 5 & 11\end{bmatrix}$$

TRANSPOSE OF A MATRIX

If the rows and corresponding columns of a matrix A of order $m \times n$ are interchanged, we get a new matrix of the order $n \times m$, which is called the transpose of the matrix A. The transpose of a matrix A is denoted by A' or A^t. For example:

If $A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 5 & -6 \\ 3 & 4 & 1 \end{bmatrix}$, then the transpose of A can be expressed as

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 5 & 4 \\ 2 & -6 & 1 \end{bmatrix}$$

Similarly, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 7 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 5 \end{bmatrix}$.

PROPERTIES OF TRANSPOSE OF A MATRIX

We now give some properties of transposed of a matrix. The proof of these properties is beyond of the scope of this book.

(i) The transpose of the sum of two matrices is the sum of their transpose, i.e., $(A + B)' = A' + B'$

(ii) The transpose of the transpose of a matrix is the matrix itself, i.e., $(A')' = A$

(iii) If A is a matrix of any order and k is a scalar, then $(kA)' = kA'$

(iv) The transpose of the product of two matrices is the product in reverse order of their transpose, i.e., $(AB)' = B'A'$

It is clear that while forming the transpose of a matrix we simply change the rows to corresponding columns and vice-versa. After having the concept of transpose matrix, try to find out the answer of the following question:

Is it possible that $A = A'$, that is, is it possible that a matrix equals its transpose?

Consider the following matrix:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 1 \end{bmatrix}$, and find out its transpose matrix.

Obviously, the transpose of A can be written as

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

Note that $A = A'$. We have thus find out a matrix A which equals its transpose. There may be a number of other matrices also which satisfy the above property. There is a special category of matrices which satisfy the above property.

DETERMINANT OF A MATRIX

If a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then determinant of matrix A is written by placing the elements of matrix A between to parallel line segments as follows:

Determinant of matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Determinant of the matrix A is denoted by $|A|$

$$\therefore \quad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The above determinant can be also written as $|a_{ij}|_{3\times 3}$, where 3×3 is the order of the determinant $|A|$.

Determinant can be find out only of square matrices. Hence order of determinant is always of the form $m \times m$, where m is a natural number. In this chapter, we will studies only 1×1, 2×2 and 3×3 determinants.

MINOR OF AN ELEMENT OF A DETERMINANT

Let $|a_{ij}|$ be a determinant of order n. Minor of any element a_{ij} of a determinant is denoted by M_{ij}. The minor M_{ij} of a_{ij} in the above given determinant, is the determinant of order $(n - 1)$ obtained by leaving the i^{th} row and j^{th} column of $|A|$.

For example, if $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is a determinant of order 2, then

M_{11} = Minor of $a_{11} = |a_{22}|$

M_{12} = Minor of $a_{12} = |a_{21}|$

M_{21} = Minor of $a_{21} = |a_{12}|$

M_{22} = Minor of $a_{22} = |a_{11}|$

If $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is a determinant of order 3, then its minors can be determined as follows:

M_{11} = Minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

M_{12} = Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

COFACTOR OF AN ELEMENT OF A DETERMINANT

Cofactor of an element a_{ij} of a determinant is denoted by A_{ij} or C_{ij}. Suppose that $|a_{ij}|$ be a determinant of order n. Then the cofactor A_{ij} of a_{ij} in the given determinant is $(-1)^{i+j}$ times the determinant of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of $|a_{ij}|$. In other word cofactor A_{ij} of a_{ij} is $(-1)^{i+j}$ times the minor M_{ij}.

Thus, A_{ij} = cofactor of $a_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}.

If $\begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$ is a determinant, then

$$C_{11}=(-1)^{1+1}M_{11}=(-1)^{1+1}\begin{vmatrix}1&2\\2&4\end{vmatrix}$$

$$C_{12}=(-1)^{1+2}M_{12}=(-1)^{1+2}\begin{vmatrix}0&2\\3&4\end{vmatrix}$$

Other cofactors can be calculated in similar manner.

VALUE OF A DETERMINANT

VALUE OF A DETERMINANT OF ORDER 1 (OR 1 × 1)

If the order of a determinant is 1, that is it has only one element. In that case, the value of the determinant is equal to the element of the matrix.

For example, $|a|=a$

VALUE OF A DETERMINANT OF ORDER 2 (OR 2 ×2)

Value of a determinant of order can be find out as follows:

$$\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}=a_{11}.a_{22}-a_{21}.a_{12}$$

= product of elements of main diagonal – product of elements of the other diagonal

For example $\begin{vmatrix}2&3\\1&4\end{vmatrix}=2\times4-1\times3=8-3=5.$

VALUE OF A DETERMINANT OF ORDER 3 (OR 3 × 3)

Value of a determinant of order 3 can be find out by expanding it along any one row or column of the determinant as given below:

$$\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} \quad \text{(expanding along first row)}$$

$$=a_{11}(-1)^{1+1}M_{11}+a_{12}(-1)^{1+2}M_{12}+a_{13}(-1)^{1+3}M_{13}$$

$$=a_{11}M_{11}-a_{12}M_{12}+a_{13}M_{13}$$

$$=a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}$$

$$=a_{11}(a_{22}\cdot a_{23}-a_{32}\cdot a_{23})-a_{12}(a_{21}\cdot a_{33}-a_{31}\cdot a_{23})+a_{13}(a_{21}\cdot a_{32}-a_{31}\cdot a_{22})$$

$$=a_{11}a_{22}a_{23}-a_{11}a_{32}a_{23}-a_{12}a_{21}a_{33}+a_{12}a_{31}a_{23}+a_{13}a_{21}a_{32}-a_{13}a_{31}a_{22}$$

OR

$$\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=a_{12}A_{12}+a_{22}A_{22}+a_{32}A_{32} \quad \text{(expanding along second column)}$$

$$=a_{12}(-1)^{1+2}M_{12}+a_{22}(-1)^{2+2}M_{22}+a_{32}(-1)^{3+2}M_{32}$$

$$=-a_{12}M_{12}+a_{22}M_{22}-a_{32}M_{32}$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}(a_{21} \cdot a_{33} - a_{31} \cdot a_{23}) + a_{22}(a_{11} \cdot a_{33} - a_{31} \cdot a_{13}) - a_{32}(a_{11} \cdot a_{23} - a_{21} \cdot a_{13})$$

$$= -a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13} - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13}$$

In the same way, we can find the value of the determinant by expanding it in any other row or column.

Illustration 2 : Evaluate the value of the determinant given by $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ 1 & 4 & 2 \end{vmatrix}$.

SOLUTION : By using the formula described above, we get

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ 1 & 4 & 2 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13}$$

$$= (-1)^{1+1} \times 1 \times \begin{vmatrix} 5 & 0 \\ 4 & 2 \end{vmatrix} + (-1)^{1+2}\,2 \times \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 0 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix}$$

$$= 5 \times 2 - 4 \times 0 - 2 \times (3 \times 2 - 1 \times 0) - (3 \times 4 - 1 \times 5)$$

$$= 10 - 12 - 7 = -9.$$

In the above mentioned method, we find the value of the determinant of order 3 × 3 by expanding the determinant along the first row. However, we can find the value of any 3 × 3 determinant by expanding the determinant along any row or column. For example, if we expand the above determinant along the first column, we get

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ 1 & 4 & 2 \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= 1\begin{vmatrix} 5 & 0 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix}$$

$$= (5 \times 2 - 4 \times 0) - 3(2 \times 2 + 4 \times 1) + (2 \times 0 + 1 \times 5)$$

$$= 10 - 24 + 5 = -9.$$

The answer is same as above. You can try by expanding the determinants along any other row or column. Every time you will get the same result unless the determinant remains unchanged.

APPLICATION OF DETERMINANTS

(i) AREA OF A TRIANGLE

Using the determinants, we can find out the area of any triangle if co-ordinate of its vertices are known. If the coordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle having these vertices

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_2 & 1 \\ x_2 & z_2 & 1 \end{vmatrix}$$

Illustration 3 : If the vertices of a triangle are (1, 2), (3, 4) and (0, 5). Find out the area of the triangle.

SOLUTION :

The area of the triangle having the given vertices

$$=\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix}$$

$$=[a_{11}c_{11}+a_{12}c_{12}+a_{13}c_{13}]$$

$$=\frac{1}{2}\left[a_{11}(-1)^{1+1}M_{11}+a_{12}(-1)^{1+2}M_{12}+a_{13}(-1)^{1+3}M_{13}\right]$$

$$=\frac{1}{2}\left[(-1)^{1+1}\times 1\times\begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix}+(-1)^{1+2}\times 2\times\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}+(-1)^{1+3}\times 1\times\begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix}\right]$$

$$=\frac{1}{2}[1(4-5)-2(3-0)+1(15-0)]$$

$$=\frac{1}{2}[-1-6+15] \;=4 \text{ square unit.}$$

(ii) CONDITION OF COLLINEARITY

If coordinates of three points are known then using the determinant, we can find that the given three points are collinear or not.

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

Illustration 4 : Check whether points $A(2, 0)$, $B(1, 3)$ and $C(0, 1)$ are collinear or not.

SOLUTION :

$$\begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix}=2\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}=6+1=7\neq 0$$

Hence the three given points are not collinear.

MISCELLANEOUS

Solved Examples

Example 1: Find X and Y if $X+Y=\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

SOLUTION: We have,

$X+Y=\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

$\therefore \quad (X+Y)+(X-Y)=\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}+\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

$\Rightarrow \quad 2X=\begin{bmatrix}7+3 & 0+0\\2+0 & 5+3\end{bmatrix}$

$\Rightarrow \quad 2X=\begin{bmatrix}10 & 0\\2 & 8\end{bmatrix} \Rightarrow X=\frac{1}{2}\begin{bmatrix}10 & 0\\2 & 8\end{bmatrix}=\begin{bmatrix}5 & 0\\1 & 4\end{bmatrix}$

and, $\quad (X+Y)-(X-Y)=\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}+\begin{bmatrix}-3 & 0\\0 & -3\end{bmatrix}$

$\Rightarrow \quad 2Y=\begin{bmatrix}7-3 & 0+0\\2+0 & 5-3\end{bmatrix}$

$\Rightarrow \quad 2Y=\begin{bmatrix}4 & 0\\2 & 2\end{bmatrix} \Rightarrow Y=\frac{1}{2}\begin{bmatrix}4 & 0\\2 & 2\end{bmatrix}=\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$

Thus, $X=\begin{bmatrix}5 & 0\\1 & 4\end{bmatrix}$ and $Y=\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$

Example 2: Find a matrix A such that $2A-3B+5C=0$, where

$B=\begin{bmatrix}-2 & 2 & 0\\3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\7 & 1 & 6\end{bmatrix}$

SOLUTION: We have,

$2A-3B+5C=0$

$\Rightarrow \quad 2A=3B-5C$

$\Rightarrow \quad 2A=3\begin{bmatrix}-2 & 2 & 0\\3 & 1 & 4\end{bmatrix}-5\begin{bmatrix}2 & 0 & -2\\7 & 1 & 6\end{bmatrix}$

$$\Rightarrow \quad 2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} -10 & 0 & 10 \\ -35 & -5 & -30 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} -6-10 & 6+0 & 0+10 \\ 9-35 & 3-5 & 12-30 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

Example 3 : **Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$. Find AB and BA and show that $AB \neq BA$.**

SOLUTION: Here, A is a 2×3 matrix and B is a 3×2 matrix. So, AB exists and it is of order 2×2.

We have, $AB = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$

$$\Rightarrow \quad AB = \begin{bmatrix} 2+2+12 & 3-4-15 \\ 6-2-4 & 9+4+5 \end{bmatrix} = \begin{bmatrix} 16 & -16 \\ 0 & 18 \end{bmatrix}$$

Again, B is a 3×2 matrix and A is a 2×3 matrix. So BA exists and it is or order 3×3.

Now, $BA = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$

$$\Rightarrow \quad BA = \begin{bmatrix} 2+9 & -4+6 & 6-3 \\ -1+6 & 2+4 & -3-2 \\ 4-15 & -8-10 & 12+5 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 3 \\ 5 & 6 & -5 \\ -11 & -18 & 17 \end{bmatrix}$$

Clearly, $AB \neq BA$.

Example 4 : **If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, find the values of α for which $A^2 = B$.**

SOLUTION : We have,

$$A^2 = B$$

$$\Rightarrow \quad \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^2+1 & 0+0 \\ \alpha+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow \quad \alpha^2 = 1$ and $\alpha + 1 = 5$

$\Rightarrow \quad \alpha = \pm 1$ and $\alpha = 4$, which is not possible.

Hence, there is no value of α for which $A^2 = B$ is true.

Example 5 : **If** $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ **and** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **then find k so that $A^2 = 8A + kI$.**

SOLUTION: We have,

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

and, $$8A + kI = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$\therefore \quad A^2 = 8A + kI$

$$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$\Rightarrow \quad 1 = 8 + k$ and $56 + k = 49 \Rightarrow k = -7$.

Example 6 : **If** $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ **and** $B = [-2 \ -1 \ -4]$, **verify that** $(AB)^T = B^T A^T$.

SOLUTION : We have,

$$A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } B = [-2 -1 -4]$$

$$\therefore \quad AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} [-2 -1 -4]$$

$$\Rightarrow \quad AB = \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$$

$$\Rightarrow \quad (AB)^T = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \qquad \text{...(i)}$$

Also, $B^T A^T = [-2 -1 -4]^T \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} [-1 \ 2 \ 3] = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix}$ ...(ii)

From (i) and (ii), we observe that

$(AB)^T = B^T A^T$

Example 7 : **Evaluate the determinant $D = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ by expanding it along first column.**

SOLUTION : By definition, we have

$$D = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

$$\Rightarrow \quad D = (-1)^{1+1} \times 2 \times \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} + (-1)^{2+1} \times 1 \times \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + (-1)^{3+1} \times (-2) \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}$$

$$\Rightarrow \quad D = 2\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}$$

$$\Rightarrow \quad D = 2(-6-3) - (-9+2) - 2(9+4) = -18 + 7 - 26 = -37.$$

Example 8 : **If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, find the determinant of the matrix $A^2 - 2A$.**

SOLUTION : We have,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad A^2 - 2A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^2 - 2A = \begin{bmatrix} 1+6 & 3+3 \\ 2+2 & 6+1 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^2 - 2A = \begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7-2 & 6-6 \\ 4-4 & 7-2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\therefore \quad |A^2 - 2A| = \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} = 25 - 0 = 25.$$

Example 9 : Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

SOLUTION : We have,

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$\Rightarrow \quad \Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = x(-x^2-1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$$

$$\Rightarrow \quad \Delta = -x^3 - x + x(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow \quad \Delta = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$\Rightarrow \quad \Delta = -x^3 - x + x = -x^3$, which is independent of θ.

Example 10 : If $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$, find the values of x.

SOLUTION : We have

$$\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$$

$$\Rightarrow \quad (x-2) \times 2x - (-3) \times 3x = 3$$

$$\Rightarrow \quad 2x(x-2) + 9x = 3$$

$$\Rightarrow \quad 2x^2 - 4x + 9x = 3$$

$$\Rightarrow \quad 2x^2 + 5x - 3 = 0$$

$$\Rightarrow \quad (2x-1)(x+3) = 0$$

$$\Rightarrow \quad 2x - 1 = 0 \text{ or, } x + 3 = 0$$

$$\Rightarrow \quad x = \frac{1}{2}, -3.$$

EXERCISE 1

FIB Fill in the Blanks

DIRECTIONS: *Complete the following statements with an appropriate word / term to be filled in the blank space(s).*

1. A matrix, in which there is only one row, is called a .... matrix.
2. A matrix, in which there is only one column, is called a .... matrix.
3. A matrix, in which the number of columns is not equal to the number of rows, is called a .... matrix.
4. A matrix in which the number of columns is equal to the number of rows, is called .... matrix.
5. A square matrix in which all elements except the diagonal elements are zero is known as .... matrix.

T/F True / False

DIRECTIONS: *Read the following statements and write your answer as true or false.*

1. A square matrix, in which all the elements in the diagonal are unity and all other elements are zero, is called a unit matrix.
2. Unit matrix is also called identity matrix.
3. A square matrix which is both upper and lower triangular is called a diagonal matrix.
4. $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ is a diagonal matrix of order 1.
5. Matrix multiplication is not commutative i.e. $AB \neq BA$.
6. All the diagonal element of skew-symmetric matrix are 1.

MTC Match the Columns

DIRECTIONS: *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D, E) in Column I have to be matched with statements (p, q, r, s, t) in column II.*

1.

Column I	Column II
(A) If $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ then the value of $\|A\|$ is	(p) -13
(B) If $\begin{vmatrix} 2 & -4 \\ 9 & d-3 \end{vmatrix} = 4$ then the value of d is	(q) $A' = -A$
(C) A square matrix A is symmetric if	(r) 0
(D) A square matrix A is skew-symmetric if	(s) $\frac{47}{2}$
(E) Area of triangle with vertices at the points (2, 7), (1,1), (10,8) is	(t) $A' = A$

SAQ Short Answer Questions

DIRECTIONS: *Give answer in 2-3 sentences.*

1. If $A = \begin{pmatrix} 7 & 2 \\ -3 & 9 \end{pmatrix}$, $B = \begin{pmatrix} p & 2 \\ -3 & 5 \end{pmatrix}$ and $AB = BA$ then find the value of p.
2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find $2A - B$.
3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that $AB \neq BA$.
4. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \;\; 3 \;\; -6]$, verify that $(AB)' = B'A'$.
5. Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.
6. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.
7. Find minors and cofactors of all the elements of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

EXERCISE 2

MCQ Multiple Choice Question

DIRECTIONS: *This section contains multiple choice questions. Each questions has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

1. If $\begin{vmatrix} 7a-5b & 3c \\ -1 & 2 \end{vmatrix} = 0$, then which one of the following is correct?

   (a) $14a + 3c = 5b$ (b) $14a - 3c = 5b$

   (c) $14a + 3c = 10b$ (d) $14a + 10b = 3c$

2. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$, then the determinant value of AB is

   (a) 10 (b) 20

   (c) 12 (d) 15

3. If $A = \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and $A = A^T$, then

   (a) $x = 0,\ y = 5$ (b) $x + y = 5$

   (c) $x = y$ (d) none of these

4. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and C_{ij} is cofactor of a_{ij} in A, then the value of $|A|$ is given by

   (a) $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$

   (b) $a_{11}C_{11} + a_{12}C_{21} + a_{13}C_{31}$

   (c) $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$

   (d) $a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

5. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is

   (a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

MTOC More than One Correct

DIRECTIONS: *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which ONE or MORE MAY be correct.*

1. Which one of the following does not hold for matrix multiplication?

   (a) Matrix multiplication is not commutative.

   (b) It is not associative.

   (c) It is not distributive over matrix addition.

   (d) The product of a matrix with a null matrix is a null matrix

2. Which of the following is/are correct?

   (a) If A and B are two matrices of the same order, then $A - B = A + (-B)$.

   (b) A square matrix $A = [a_{ij}]$ is called an upper triangular matrix, if $a_{ij} = 0$ for all $i > j$.

   (c) Addition of two matrices is commutative if A and B are two matrices of the same order then $A + B = B + A$.

   (d) A matrix whose all elements are zero is called an identity matrix.

3. Which one of the following is/are not correct?

   (a) If the order of a matrix is $m \times n$, then order of transpose of the matrix is same.

   (b) If the order of a matrix is $m \times n$, then order of transpose of the matrix is $n \times m$.

   (c) If A and B are two matrices of same order, then $(A + B)^T = A^T + B^T$.

   (d) If A and B are two matrices of same order, then $(A + B)^T = A + B$.

4. Which one of the following is/are correct?

   (a) Matrix subtraction is possible only when both the matrices are of same order.

   (b) $A - B$, is obtained by subtracting corresponding element of B from that of A.

   (c) If 'k' is a scalar and A and B are two matrices of same order, then $k(A + B) = kA = kB$.

   (d) None of the above.

PBQ Passage Based Questions

DIRECTIONS: *Study the given paragraph(s) and answer the following questions.*

A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non-singular matrix.

1. For what value of x the matrix is $A=\begin{bmatrix}1 & -2 & 3\\ 1 & 2 & 1\\ x & 2 & -3\end{bmatrix}$ is singular?

(a) $x=-1$ (b) $x=1$

(c) $x=2$ (d) $x=-2$

2. If A is a skew-symmetric matrix of odd order then

(a) A is singular (b) A is non-singular

(c) cannot determined (d) none of these

3. If A is a square matrix of order $n(n \geq 2)$ such that each element in a row (column) of A is zero then

(a) $|A|=0$ (b) $|A|\neq 0$

(c) $|A|\pm 1$ (d) none

A&R Assertion & Reason

DIRECTIONS: *Each of these questions contains an Assertion followed by reason. Read them carefully and answer the question on the basis of following options. You have to select the one that best describes the two statements.*

(a) If both **Assertion** and **Reason** are **correct** and Reason is the **correct explanation** of Assertion.

(b) If both **Assertion** and **Reason** are correct, but Reason **is not the correct explanation** of Assertion.

(c) If **Assertion** is **correct** but **Reason** is **incorrect.**

(d) If **Assertion** is **incorrect** but **Reason** is **correct.**

1. **Assertion :** $\begin{bmatrix}3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 7\end{bmatrix}$ is a diagonal matrix.

Reason : $A=[a_{ij}]$ is square matrix such that $a_{ij}=0\ \forall i \neq j$ then A is called diagonal matrix.

2. **Assertion :** The order of the matrix A is 3×5 and that of B is 2×3. Then the matrix AB is not possible.

Reason : No. of columns in A is not equal to no. of rows in B.

3. **Assertion :** An $m \times n$ matrix is said to be a square matrix if $m=n$ and is known as a square matrix of order n.

Reason : A matrix having m rows and n columns is called a matrix of order $m \times n$.

4. **Assertion :** Matrix $A=\begin{bmatrix}2 & 4 & 5\\ -1 & 8 & 7\end{bmatrix}$ is called a rectangular matrix.

Reason : A matrix, in which the number of columns is not equal to the number of rows, is called a rectangular matrix.

HOTS Hots Subjective Questions

DIRECTIONS: *Answer the following questions.*

1. If $A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$, then find the value of A^{40}.

2. If A and B are 3×3 matrices such that $AB=A$ and $BA=B$, then find the value of A^3-B^3.

3. Under what conditions the matrix equation $A^2-B^2=(A-B)(A+B)$ is true?

4. If A, B, and C are three matrices of the same order, then prove that $A=B \Rightarrow A+C=B+C$.

5. If $A=\begin{bmatrix}0 & -\tan(\alpha/2)\\ \tan(\alpha/2) & 0\end{bmatrix}$ and I be the identity matrix of order 2, then show that

$$I+A=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

6. If $A=\begin{bmatrix}2 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1\\ 2 & 2\\ 1 & 3\end{bmatrix}$. Write the order of AB and BA.

7. If $A=\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}$ and $A^n=0$, then find the minimum value of n.

8. Find all 2×2 matrices which commute with the matrix $A=\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}$.

SOLUTIONS

EXERCISE – I

FILL IN THE BLANKS

1. Row
2. Column
3. Rectangular
4. Square
5. Diagonal

TRUE/FALSE

1. True
2. True
3. True
4. False

$\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ is a diagonal matrix of order 2.

5. True
6. False

MATCH THE COLUMNS

1. (A) → r; (B) → p; (C) → t; (D) → q; (E) → s

(A) $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

$\Rightarrow\ |A| = 2(9) - 3(6) = 18 - 18 = 0$

(B) $\begin{vmatrix} 2 & -4 \\ 9 & d-3 \end{vmatrix} = 4$

$\Rightarrow\ 2(d-3) + 4(9) = 4$

$\Rightarrow\ 2d - 6 + 36 = 4$

$\Rightarrow\ 2d = -26 \Rightarrow d = -13$

(E) $\Delta = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

$= \frac{1}{2}[2(1-8) - 7(1-10) + 1(8-10)]$

$= \frac{1}{2}[-14 + 63 - 2] = \frac{1}{2}[47]$ sq. unit

SHORT ANSWER QUESTIONS

1. $AB = \begin{pmatrix} 7 & 2 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} p & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 7p-6 & 24 \\ -3p-27 & 39 \end{pmatrix}$

$BA = \begin{pmatrix} p & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ -3 & 9 \end{pmatrix} = \begin{pmatrix} 7p-6 & 2p+18 \\ -21-15 & -6+45 \end{pmatrix}$

$AB = BA$

$\Rightarrow\ 2p + 18 = 24 \Rightarrow p = +3$

2. We have

$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$

$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$

3. Since A is a 2×3 matrix and B is 3×2 matrix. Hence AB and BA are both defined and are matrices of order 2×2 and 3×3, respectively.

$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$

and

$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix}$

$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$

Clearly $AB \neq BA$

4. We have,

$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = [1 \quad 3 \quad -6]$

then $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \quad 3 \quad -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

$$(AB)' = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$$

Now $A' = [-2\ 4\ 5]$, $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \quad 4 \quad 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

5. We have $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

i.e., $3 - x^2 = 3 - 8$

i.e., $x^2 = 8$

Hence, $x = \pm 2\sqrt{2}$

6. We have,

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$\therefore \quad |A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 2 - 8 = -6$ and $|2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$

$= 8 - 32 = -24$

Consider $|2A| = -24 = 4 \times -6 = 4|A|$ Hence proved.

7. Minor of the element a_{ij} is M_{ij}.

Here $a_{11} = 1$. So M_{11} = Minor of $a_{11} = 3$

M_{12} = Minor of the element $a_{12} = 4$

M_{21} = Minor of the element $a_{21} = -2$

M_{22} = Minor of the element $a_{22} = 1$

Now, cofactor of a_{ij} is A_{ij}. So

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$

EXERCISE – 2

MULTIPLE CHOICE QUESTIONS

1. (c) We know $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\therefore$ Req. det $= 2(7a - 5b) + 3c = 0$

2. (b) det (AB) = det$A$. det$B$.

det $A = 6 + 4 = 10$

det $B = -10 + 12 = 2$

$\therefore$ det $(AB) = 10 \times 2 = 20$

3. (c) $A = A^T$

$\Rightarrow \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix} = \begin{bmatrix} 5 & y \\ x & 0 \end{bmatrix}$

$\Rightarrow x = y$

4. (d)

5. (c) $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix} \Rightarrow |A^3| = |A|^3 = 125$

Now $|A| = \alpha^2 - 4$

$\Rightarrow (\alpha^2 - 4)^3 = 125 = 5^3 \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$

MORE THAN ONE CORRECT

1. (b, c)
2. (a, b, c)
3. (a, d)
4. (a, b, c)

PASSAGE BASED QUESTIONS

1. (a) $|A| = 0$

$\Rightarrow \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$

$\Rightarrow (-6 - 2) + 2(-3 - x) + 3(2 - 2x) = 0$

$\Rightarrow -8x - 8 = 0$

$\Rightarrow x = -1$

2. (a) A is singular
3. (a)

ASSERTION & REASON

1. (a) If $A = [a_{ij}]_{n \times n}$ is a square matrix such that $a_{ij} = 0$ for i, j then A is called diagonal matrix the above statement is true and $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix.

2. (a) Matrix AB is possible only when if number of columns in A is equal to number of rows in B.
3. (b) Both statements are true.
4. (a) Both Assertion and Reason is correct. Also, Reason is correct explanation for Assertion.

HOTS SUBJECTIVE QUESTIONS

1. $A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$, $A^2\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$,

$\Rightarrow A^{40}=(A^2)^{20}=(I)^{20}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

2. Given $AB=A$, $\therefore\ B=I$

$BA=B$, $\therefore A=I$

Now, $A^3-B^3=(I)^3-(I)^3=O$.

3. We have, $A^2-B^2=(A-B)(A+B)$

$\Rightarrow\ A^2-B^2=(A-B)A+(A-B)B$

[By distributive property of matrix multiplication over ~~matrix~~ addition]

$\Rightarrow A^2-B^2=A^2-BA+AB-B^2$

[By distributive property of matrix multiplication over matrix subtraction]

$\Rightarrow\ 0=A^2-BA+AB-B^2-A^2+B^2$

$\Rightarrow\ 0=-BA+AB\ \Rightarrow AB=BA$

Thus the given matrix equation is true if the matrices A and B commute with each other over multiplication.

4. Let A $=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ and $C=[c_{ij}]_{m\times n}$ be three matrices of the same order $m\times n$. Then, $A+C$ and $B+C$ are also of order $m\times n$.

Now, $A=B$

$\Rightarrow\ a_{ij}=b_{ij}$ for all $i=1,2,\ldots,m;\ j=1,2,\ldots,n$

$\Rightarrow\ a_{ij}+c_{ij}=b_{ij}+c_{ij}$ for all $i=1,2,\ldots,m; j=1,2,\ldots,n$

$\Rightarrow\ (A+C)_{ij}=(B+C)_{ij}$ for all $i=m;\ j=n$

$\Rightarrow\ A+C=B+C$.

5. We have,

$$I+A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$$

And $$I-A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}$$

$$\therefore\ (I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}\begin{bmatrix}\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & -\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\\ \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\end{bmatrix}$$

$$=\begin{bmatrix}1 & t\\ -t & 1\end{bmatrix}\begin{bmatrix}\dfrac{1-t^2}{1+t^2} & -\dfrac{2t}{1+t^2}\\ \dfrac{2t}{1+t^2} & \dfrac{1-t^2}{1+t^2}\end{bmatrix},\ \text{where } t=\tan\frac{\alpha}{2}$$

$$=\begin{bmatrix}\dfrac{1-t^2+2t^2}{1+t^2} & \dfrac{-2t+t-t^3}{1+t^2}\\ \dfrac{-t+t^3+2t}{1+t^2} & \dfrac{2t^2+1-t^2}{1+t^2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1+t^2}{1+t^2} & \dfrac{-t(1+t^2)}{1+t^2}\\ \dfrac{t(1+t^2)}{1+t^2} & \dfrac{1+t^2}{1+t^2}\end{bmatrix}=\begin{bmatrix}1 & -t\\ t & 1\end{bmatrix}$$

$$\Rightarrow\ (I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}=I+A$$

6. Order of $A=2\times 3$

Order of $B=3\times 2$

$\therefore$ Order of AB = (Number of rows of A) $\times$ (Number of columns of B) $=2\times 2$

Again order of $B=3\times 2$ and order of $A=2\times 3$

$\therefore$ Order of $BA=3\times 3$.

7. $$A^2=A.A=\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}$$

$$=\begin{bmatrix}a^2b^2-a^2b^2 & ab^3-ab^3\\ -a^3b+a^3b & -a^2b^2+a^2b^2\end{bmatrix}=0$$

Now, $A^3=A.A^2=0$ and $A^n=0$, for all $n\geq 2$.

8. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix which commute with A then $AB = BA$.

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

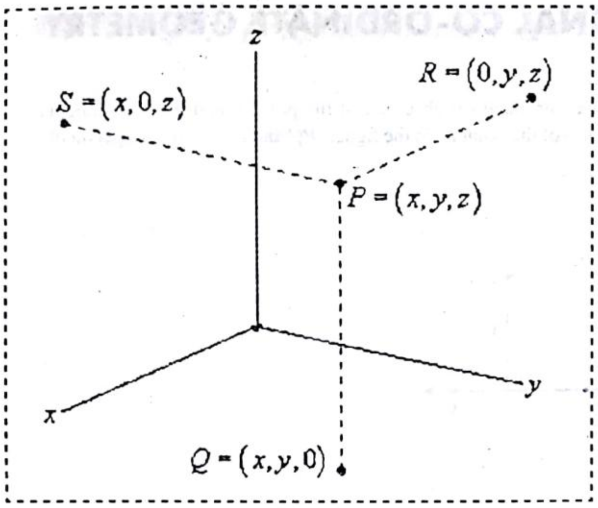
$$= \begin{bmatrix} a+2c & b+2d \\ -a-c & -b-d \end{bmatrix} = \begin{bmatrix} a-b & 2a-b \\ c-d & 2c-d \end{bmatrix}$$

$\Rightarrow a + 2c = a - b,\ b + 2d = 2a - b,\ -a - c = c - d,\ -b - d = 2c - d$

The above four relations are equivalent to only two independent relations $a - d = b$, $b + 2c = 0$

if $d = \lambda$ then $a = b + \lambda = -2c + \lambda$

Thus, $\begin{bmatrix} \lambda - 2c & -2c \\ c & \lambda \end{bmatrix}$ are all possible 2×2 matrices which commute with given matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, where λ and c being any arbitrary complex numbers.



16

CHAPTER

Extension of 2-D Geoemtry to 3-Geometry

INTRODUCTION

To locate the position of a point say point P in a plane, we consider an arbitrary point 'O', called origin, and two mutually perpendicular lines passing through the origin in the plane of the point P. One of these two lines is horizontal and other is vertical. The horizontal line is represented by X'OX and vertical line is represented by YOY'. The horizontal line X'OX is called x-axis and vertical line YOY' is called y-axis. The plane passing through these x and y-axis is called XOY plane or XY-plane.

The perpendicular distance of the point P from y-axis is called x-co-ordinate of the point P and the perpendicular distance of the point P from x-axis is called y-co-ordinate of the point P. If 'a' and 'b' are x and y-co-ordinate of the point P, then (a, b) is called coordinates of the point P, which represent the position of the point P in xy-plane.

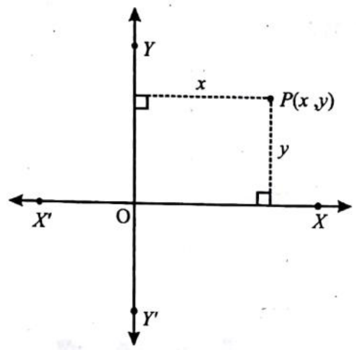
In (a, b), the x-co-ordinate 'a' always comes first and y-coordinate 'b' always comes after x-coordinate 'a'. Direction of rays OX and OY are the +ve direction of x and y-axis respectively. Direction of rays OX' and OY' are the –ve direction of x and y-axis respectively.

In actual life, we do not have to deal with the points lying in a plane only but we also deal with the points lie in the space In this chapter, we start with the points lie in a plane (a plane has two dimensions : length and breadth) and extends our study to points in space (a space has three dimensions : length, breadth and height).

INTRODUCTION TO 2-DIMENTIONAL CO-ORDINATE GEOMETRY

CO-ORDINATE OF A POINT IN xy-PLANE

The perpendicular distance of any point P from y-axis is called x-co-ordinate (or abscissa) of the point P and the perpendicular distance of the point P from x-axis is called y-co-ordinate (or ordinate) of the point P. In the figure, PQ and PR are the perpendicular distances of the point P from y and x-axis.



If $PQ = x$ units ($= OR$), $PR = y$ units ($= OQ$), the position of point P is represented by $P(x, y)$. Here the order pair (x, y) is called the co-ordinates of point P. Hence, $P(x, y)$ is a point in the x, y-plane which is x unit distant from y-axis and y units distant from x-axis.

In the order pair (x, y), the first entry 'x' is the x-co-ordinate and the second entry 'y' is the y-co-ordinate of the point. (x, y) is called an order pair because there is a pair of numbers x and y, in which the first entry x is always x-co-ordinate and the second entry y is always y-co-ordinate of the point.

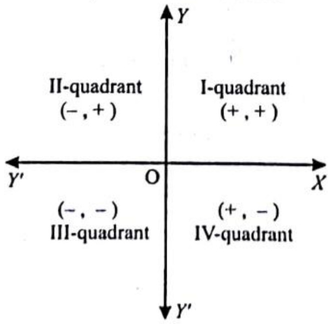
SIGN CONVENTIONS IN THE xy-PLANE

(a) All the distances are measured from origin(O).
(b) All the distances measured along or parallel to x-axis and right side of origin are taken as $+ve$.
(c) All the distances measured along or parallel to x-axis and left side of origin are taken as $-ve$.
(d) All the distances measured along or parallel to y-axis and above the origin are taken as $+ve$.
(e) All the distances measure along or parallel to y-axis and below the origin are taken as $-ve$.

According to the above sign conventions :

(i) Co-ordinate of origin is $(0, 0)$
(ii) Co-ordinate of any point on the x-axis and right side of origin is of the form $(x, 0)$, where $x > 0$.
(iii) Co-ordinate of any point on the x-axis and left side of origin is of the form $(-x, 0)$, where $x > 0$.
(iv) Co-ordinate of any point on the y-axis and above the origin is of the form $(0, y)$, where $y > 0$.
(v) Co-ordinate of any point on the y-axis and below the origin is of the form $(0, -y)$, where $y > 0$.
(vi) x and y-axis divide the XOY plane in four parts. Each part is called a quadrant.

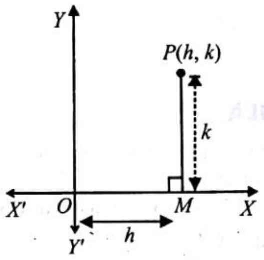
The four quadrants are written as I-quadrant (XOY), II-quadrant (YOX'), III-quadrant ($X'OY'$) and IV-quadrant ($Y'OX$). Each of these quadrants shows the specific quadrant of the XOY plane as shown below :



Note :
(a) Any of the four quadrants does not includes any part of x or y-axis.
(b) In the first quadrant both x and y-co-ordinates of any point are $+ve$.
(c) In second quadrant, x-coordinate is $-ve$ and y-coordinate is $+ve$.
(d) In third quadrant, both x and y-co-ordinates of any point are $-ve$.
(e) In fourth quadrant, x-co-ordinate is $+ve$ and y-co-ordinate is $-ve$ as shown in the above diagram.

PLOTTING OF A POINT IN THE xy-PLANE WHOSE COORDINATES ARE KNOWN

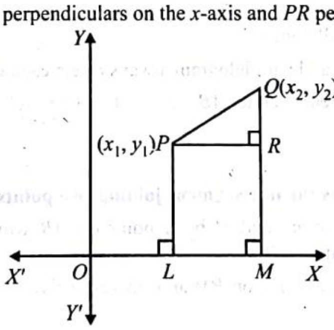
If coordinate of a point is given then the position of the point can be plotted by measuring its proper distances from both the axes. Thus, any point (h, k) can be plotted as follows :



(i) Measure OM equal to h along the x-axis.
(ii) Now measure MP perpendicular to OM equal to k.
The point P is the position of the given point (x, y).

DISTANCE BETWEEN TWO POINTS

Distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is the length of the line segment PQ.
From P and Q draw PL and QM respectively perpendiculars on the x-axis and PR perpendicular on QM.



Then, $OL = x_1$, $OM = x_2$, $PL = y_1$ and $QM = y_2$

$\therefore \quad PR = LM = OM - OL = x_2 - x_1$

$QR = QM - RM = QM - PL = y_2 - y_1$

Since PRQ is a right angled triangle,

$\therefore \quad PQ^2 = PR^2 + QR^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$ (By the Pythagoras Theorem)

$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

i.e., Distance between any two points $= \sqrt{(\text{difference of abscissae})^2 + (\text{difference of ordinates})^2}$

Corollary : The distance of the point (x_1, y_1) from the origin $(0, 0)$ is

$$\sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = \sqrt{x_1^2 + y_1^2}$$

Let us consider some examples to illustrate.

Illustration1 : Find the distance between each of the following pair of points :

(a) $P(6, 8)$ and $Q(-9, -12)$ **(b)** $A(-6, -1)$ and $B(-6, 11)$

SOLUTION :

(a) Here the points are $P(6, 8)$ and $Q(-9, -12)$.

By using distance formula, we have

$$PQ = \sqrt{(-9-6)^2 + (-12-8)^2} = \sqrt{15^2 + 20^2} = \sqrt{225+400} = \sqrt{625} = 25$$

Hence, $PQ = 25$ units.

(b) Here the points are $A(-6, -1)$ and $B(-6, 11)$

By using distance formula, we have

$$AB = \sqrt{\{-6-(-6)\}^2 + \{11-(-1)\}^2} = \sqrt{0^2 + 12^2} = 12$$

Hence, $AB = 12$ units.

APPLICATIONS OF DISTANCE FORMULA

(i) For given three points A, B, C to decide whether they are collinear or vertices of a particular triangle. First we find the length of AB, BC, and CA then we shall find that the point are

(a) Collinear, if the sum of two shorter distances is equal to the longest distance.

(b) Vertices of an equilateral triangle if $AB = BC = CA$.

(c) Vertices of an isosceles triangle if $AB = BC$ or $BC = CA$ or $CA = AB$.

(d) Vertices of a right angled triangle if $AB^2 + BC^2 = CA^2$ or $AB^2 + CA^2 = BC^2$ or $BC^2 + CA^2 = AB^2$

(ii) For given four points A, B, C and D;

(a) $AB = BC = CD = DA$; $AC = BD \Rightarrow ABCD$ is a square

(b) $AB = BC = CD = DA$; $\Rightarrow ABCD$ is a rhombus

(c) $AB = CD$, $BC = DA$; $AC = BD \Rightarrow ABCD$ is a rectangle

(d) $AB = CD$, $BC = DA \Rightarrow ABCD$ is a parallelogram

(iii) (a) Diagonal of square, rhombus, rectangle and parallelogram always bisect each other

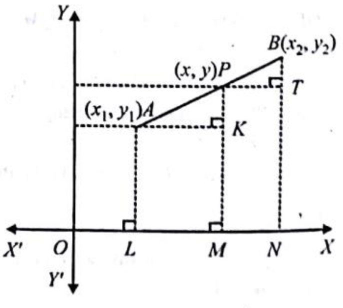
(b) Three given points A, B and C are collinear if either $AB + BC = AC$, $AB + AC = BC$ or $BC + AC = AB$

SECTION FORMULA

To find the co-ordinates of a point, which divides the line segment joining two points internally in a given ratio :

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points and P be a point on AB which divides it in the given ratio $m : n$ i.e., $AP : PB = m : n$. We have to find the co-ordinates of P. Let $P = (x, y)$.

Draw the perpendiculars AL, PM, BN on OX, and, AK, PT on PM and BN respectively. Then two similar triangles AKP and PTB are formed.



We have, $\dfrac{AP}{PB} = \dfrac{AK}{PT} = \dfrac{PK}{BT}$ .... (i)

Now, $AK = LM = OM - OL = x - x_1$

$PT = MN = ON - OM = x_2 - x$

$PK = MP - MK = MP - LA = y - y_1$

$BT = NB - NT = NB - MP = y_2 - y$

From (i), we have, $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x} = \dfrac{y - y_1}{y_2 - y}$

The first two relations give, $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x}$

or $mx_2 - mx = nx - nx_1$

or $x(m + n) = mx_2 + nx_1$

or $x = \dfrac{mx_2 + nx_1}{m + n}$

Similarly, from the relation $\frac{AP}{PB}=\frac{PK}{BT}$, we get $\frac{m}{n}=\frac{y-y_1}{y_2-y}$ which gives on simplification.

$$y=\frac{my_2+ny_1}{m+n}$$

Hence, $x=\frac{mx_2+nx_1}{m+n}$ and $y=\frac{my_2+ny_1}{m+n}$ .... (ii)

Hence co-ordinates of a point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$ internally is $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

MID-POINT FORMULA

The co-ordinates of the mid-point of the line segment joining two points (x_1, y_1) and (x_2, y_2) can be obtained by taking $m = n$ in the section formula above. Putting $m = n$ in (ii) above, we have

$$x=\frac{nx_2+nx_1}{n+n}=\frac{x_2+x_1}{2} \text{ and } y=\frac{ny_2+ny_1}{n+n}=\frac{y_2+y_1}{2}$$

Hence, co-ordinates of the mid-point joining two points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

CENTROID OF A TRIANGLE

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle, then the centroid is the point of intersection of the medians (Line segment joining the mid point of a side and its opposite vertex is called a median of the triangle). Centroid divides the median from vertex to the mid point in the ratio of 2 : 1.

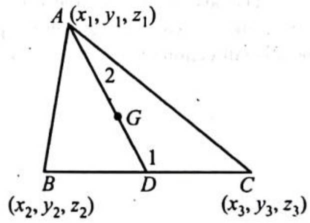
In the figure, D is the mid point of BC.

$\therefore \quad D=\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

Since the centroid G divides the median AD in the ratio 2 : 1.

$\therefore$ Co-ordinates of centroid,

$$G=\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{3}, \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{3}\right)=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$



AREA OF A TRIANGLE

Let ABC be any triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw BL, AM and CN perpendiculars from B, A and C respectively on the x-axis. $ABLM$, $AMNC$ and $BLNC$ are all trapeziums.

Area of ΔABC = Area of trapezium $ABLM$ + Area of trapezium $AMNC$ – Area of trapezium $BLNC$

We know that, area of trapezium

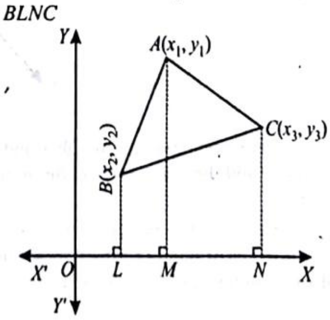
$= \frac{1}{2} \times$ (sum of parallel sides) × (distance between them)

Therefore, area of ΔABC

$$= \frac{1}{2}(BL+AM)(LM)+\frac{1}{2}(AM+CN)(MN)-\frac{1}{2}(BL+CN)(LN)$$

$$= \frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$$

$$= \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$



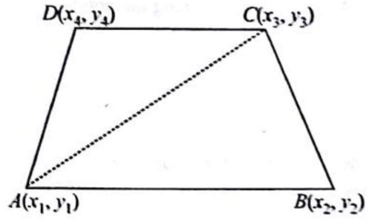
CONDITION OF COLLINEARITY OF THREE POINTS

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, if $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

AREA OF A QUADRILATERAL

Let the vertices of quadrilateral $ABCD$ are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$

So, area of quadrilateral $ABCD$ = Area of ΔABC + Area of ΔACD

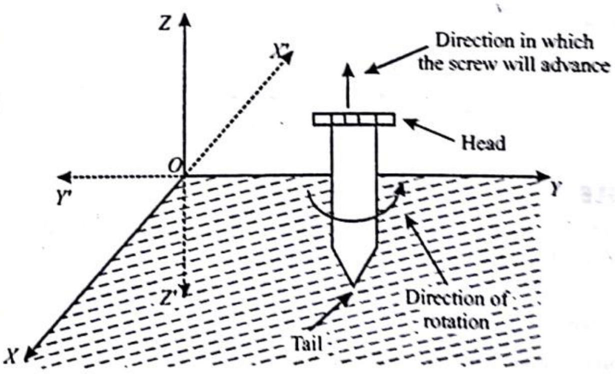


INTRODUCTION TO 3-DIMENSIONAL CO-ORDINATE GEOMETRY

TO LOCATE THE POSITION OF A POINT IN SPACE : CO-ORDINATES OF A POINT

To locate the position of a point, say point in space, we consider an arbitrary point 'O', called origin and three mutually perpendicular lines passing through the origin 'O'. One of these three lines is called x-axis and represented by $X'OX$, other one is called y-axis and represented by $Y'OY$, the remaining one is called z-axis and represented by $Z'OZ$. Direction of rays OX', OY' and OZ' are the positive directions of x, y and z-axis respectively. Direction of rays OX', OY' and OZ' are the negative direction of x, y and z-axis respectively.

The positive direction of x, y and z-axis are such that if we place a screw perpendicular to the xy-plane so that tail of the screw lies in the xy-plane in between the positive direction of x and y-axis and rotate the screw from positive direction of x-axis to the positive direction of y-axis, the screw will advance in the positive direction of z-axis as shown in the given below figure.



The plane passing through x and y-axis is called XOY or xy-plane, the plane passing through y and z-axis is called YOZ or yz-plane and the plane passing through z and x axis is called ZOX or zx-plane. These three planes are called co-ordinate planes in space.

If a, b and c are the distances of the point Q from yz-plane, zx-plane and xy-plane respectively, then (a, b, c) is called co-ordinates of point Q, which locate the position of the point Q. In (a, b, c); x-co-ordinate 'a' always comes first, y-co-ordinate 'b' always comes in the middle and z-co-ordinate 'c' always comes at the end.

SIGN OF CO-ORDINATES OF A POINTS IN SPACE

The co-ordinate planes (i.e. xy-plane, yz-plane and zx-planes) divides the whole space in eight region. Each region is called an octant. The sign of the co-ordinates of a point determine by the octant in which the point lies. The following table shows the signs of the co-ordinates in eight octants.

Octants→ / Co-ordinates ↓	First Octant ($OXYZ$)	Second Octant ($OX'YZ$)	Third Octant ($OX'Y'Z$)	Fourth Octant ($OXY'Z$)	Fifth Octant ($OXYZ'$)	Sixth Octant ($OX'YZ'$)	Seventh Octant ($OX'Y'Z'$)	Eighth Octant ($OXY'Z'$)
x-co-ordinate	+	−	−	+	+	−	−	+
y-co-ordinate	+	+	−	−	+	+	−	−
z-co-ordinate	+	+	+	+	−	−	−	−

DISTANCE BETWEEN TWO POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. The distance between these points is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The distance of a point $P(x, y, z)$ from origin O is

$$OP = \sqrt{x^2 + y^2 + z^2}$$

Illustration 2 : Show that the points (1, –1, 3), (2, –4, 5) and (5, –13, 11) are collinear.

SOLUTION : Let the points be represented by $A(1, -1, 3)$, $B(2, -4, 5)$ and $C(5, -13, 11)$. Then by using distance formula

$$AB = \sqrt{(2-1)^2 + (-4+1)^2 + (5-3)^2} = \sqrt{1+9+4} = \sqrt{14}$$

$$BC = \sqrt{(5-2)^2 + (-13+4)^2 + (11-5)^2}$$

$$= \sqrt{9+81+36} = \sqrt{126} = 3\sqrt{14}$$

$$CA = \sqrt{(1-5)^2 + (-1+13)^2 + (3-11)^2}$$

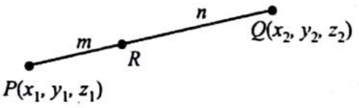
$$= \sqrt{16+144+64} = \sqrt{224} = 4\sqrt{14}$$

We note that $AB + BC = CA$. Thus, A, B, and C are collinear.

SECTION FORMULAE

The co-ordinates of any point R, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ internally are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + mz_1}{m+n}\right)$$



MID-POINT FORMULA

The co-ordinates of mid point of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

CENTROID OF A TRIANGLE

Co-ordinate of the centroid of a triangle formed with vertices $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

MISCELLANEOUS

Solved Examples

Example 1 : Find the distance between each of the following points $A(-6, -1)$ and $B(-6, 11)$

SOLUTION : Here the points are $A(-6, -1)$ and $B(-6, 11)$

By using distance formula, we have

$$AB = \sqrt{\{-6-(-6)\}^2 + \{11-(-1)\}^2} = \sqrt{0^2 + 12^2} = 12$$

Hence, $AB = 12$ units.

Example 2 : The distance between two points $(0, 0)$ and $(x, 3)$ is 5. Find x.

SOLUTION : By using distance formula, we have the distance between $(0, 0)$ and $(x, 3)$ is $\sqrt{(x-0)^2 + (3-0)^2}$.

It is given that $\sqrt{(x-0)^2 + (3-0)^2} = 5$ or $\sqrt{x^2 + 3^2} = 5$

Squaring both sides, $x^2 + 9 = 25$ or $x^2 = 16$ or $x = \pm 4$

Hence, $x = +4$ or $x = -4$

Example 3 : Show that the points $(1, 1)$, $(3, 0)$ and $(-1, 2)$ are collinear

SOLUTION : Let $P(1, 1)$, $Q(3, 0)$ and $R(-1, 2)$ be the given points

$$PQ = \sqrt{(3-1)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5} \text{ units}$$

$$QR = \sqrt{(-1-3)^2 + (2-0)^2} = \sqrt{16+4} = 2\sqrt{5} \text{ units}$$

$$RP = \sqrt{(-1-1)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5} \text{ units}$$

Now, $PQ + PR = (\sqrt{5} + \sqrt{5})$ units $= 2\sqrt{5}$ units $= QR$

$\therefore$ P, Q and R are collinear points.

Example 4 : Find the co-ordinates of a point on y-axis which is equidistant from the points $(13, 2)$ and $(12, -3)$.

SOLUTION : Let $P(0, y)$ be the required point and the given points be $A(12, -3)$ and $B(13, 2)$.

Then $PA = PB$ (given)

$$\sqrt{(12-0)^2 + (-3-y)^2} = \sqrt{(13-0)^2 + (2-y)^2}$$

$$\Rightarrow \sqrt{144 + (y+3)^2} = \sqrt{169 + (2-y)^2}$$

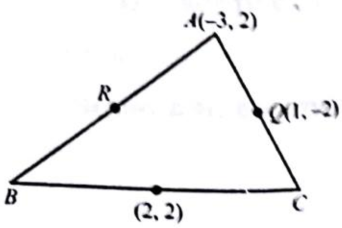
Taking square on both sides, we get

$144 + 9 + y^2 + 6y = 169 + 4 + y^2 - 4y$

$\Rightarrow$ $10y = 20 \Rightarrow y = 2$

$\therefore$ The required point on y-axis is $(0, 2)$.

Example 5 : ABC is a triangle in which P, Q, R are the mid-points of BC, CA, AB respectively. The co-ordinates are $A(-3, 2)$, $Q(1, -2)$ and $P(2, 2)$. Find $\overline{RQ}$.



SOLUTION : Let the coordinates of C be (x, y).

$$\Rightarrow \left(\frac{-3+x}{2}, \frac{2+y}{2}\right) = (1, -2)$$

$$\Rightarrow \frac{-3+x}{2} = 1, \frac{2+y}{2} = -2 \Rightarrow x = 5, y = -6$$

Let the coordinates of B be (a, b).

$\Rightarrow \quad \left(\frac{a+5}{2}, \frac{b-6}{2}\right) = (2, 2) \Rightarrow a = -1, b = 10$

Coordinates of $R = \left(\frac{-1-3}{2}, \frac{10+2}{2}\right) = (-2, 6)$

$$RQ = \sqrt{(1+2)^2 + (-2-6)^2} \Rightarrow \sqrt{9+64} = \sqrt{73}$$

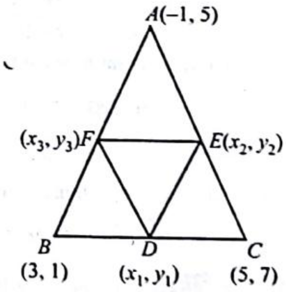
Example 6 : **Find the ratio in which the join of (– 4, 3) and (5, –2) is divided by (i) x-axis (ii) y-axis.**

SOLUTION :

(*i*) x-axis divides the join of (x_1, y_1) and (x_2, y_2) in the ratio of $-y_1 : y_2 = -3 : -2 = 3 : 2$.

(*ii*) y-axis divides, in the ratio of $-x_1 : x_2 = 4 : 5$.

Example 7 : **The co-ordinates of A, B and C are (–1, 5), (3, 1) and (5, 7) respectively. D, E and F are the middle points of BC, CA and AB respectively. Calculate the area of the triangle DEF.**



SOLUTION : Mid-point $D\,(x_1, y_1) = \left(\frac{3+5}{2}, \frac{1+7}{2}\right) = (4, 4)$

Mid-point $E\,(x_2, y_2) = \left(\frac{-1+5}{2}, \frac{5+7}{2}\right) = (2, 6)$

Mid-point $F\,(x_3, y_3) = \left(\frac{-1+3}{2}, \frac{5+1}{2}\right) = (1, 3)$

Now, using the formula, area of triangle $= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

$\Rightarrow \quad$ Area of $\Delta DEF = \frac{1}{2}\left|4(6-3) + 2(3-4) + 1(4-6)\right| = 4$ square units.

Hence, the area of ΔDEF is 4 square units.

Example 8 : **What kind of triangle is formed by $A(1, 2)$, $B(4, 3)$ and $C(5, 6)$?**

SOLUTION : $AB^2 = (4-1)^2 + (3-2)^2 = 9 + 1 = 10$

$BC^2 = (5-4)^2 + (6-3)^2 = 1 + 9 = 10$

$CA^2 = (5-1)^2 + (6-2)^2 = 16 + 16 = 32$

$AB^2 = BC^2 \quad \Rightarrow \quad$ it is isosceles.

$CA^2 > AB^2 + BC^2$ since $32 > 10 + 10 \quad \Rightarrow \quad \angle B$ is obtuse

Hence, ABC is an obtuse isosceles Δ.

Example 9 : **Find the co-ordinates of a point which divides the line segment joining each of the following points in the given ratio : (–1, 4) and (0, –3) in the ratio 1 : 4 internally.**

SOLUTION : Let A (–1, 4) and B (0, –3) be the given points and let $P(x, y)$ divide AB in the ratio 1 : 4 internally

Using section formula, we have

$x = \frac{1 \times 0 + 4 \times (-1)}{1+4} = -\frac{4}{5}$ and $y = \frac{1 \times (-3) + 4 \times 4}{1+4} = \frac{13}{5}$

$\therefore \quad P\left(-\frac{4}{5}, \frac{13}{5}\right)$ divides AB in the ratio 1 : 4 internally.

Example 10 : **Find the distance between the points $P(3, 4, 5)$ and $Q(-1, 2, -3)$.**

SOLUTION : $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Here, $(x_1, y_1, z_1) \equiv (3, 4, 5)$ and $(x_2, y_2, z_2) \equiv (-1, 2, -3)$

Hence, $PQ = \sqrt{(-1-3)^2 + (2-4)^2 + (-3-5)^2} = \sqrt{16+4+64} = \sqrt{84} = 2\sqrt{21}$

Example 11 : **Show that the points $A(0, 7, 10)$, $B(-1, 6, 6)$ and $C(-4, 9, 6)$ are vertices of an isosceles right-angled triangle.**

SOLUTION : $AB^2 = (-1-0)^2 + (6-7)^2 + (6-10)^2 = 18$

$BC^2 = (-1+4)^2 + (6-9)^2 + (6-6)^2 = 18$

$CA^2 = (0+4)^2 + (7-9)^2 + (10-6)^2 = 36$

Since $AB = BC$ and $AB^2 + BC^2 = AC^2$.

Therefore A, B and C are vertices of an isosceles right-angled triangle.

Example 12 : Find the ratio in which the planes (1) XY (2) YZ divide the line joining the points $P(-2, 4, 7)$ and $Q(3, -5, 8)$.

SOLUTION :

(1) Let the xy-plane divide the line joining the points $P(-2, 4, 7)$ and $Q(3, -5, 8)$ in the ratio $\lambda : 1$ at point R. Then the co-ordinates of R are $\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right)$.

Since, the point R lies in the xy-plane, its z-co-ordinate $= 0$

$\Rightarrow \frac{8\lambda+7}{\lambda+1} = 0 \Rightarrow 8\lambda + 7 = 0$ or $\lambda = -\frac{7}{8}$

Hence, the required ratio is 7 : 8. Also, here $\lambda < 0$, so the xy-plane divides the line segment PQ externally.

(2) Here, we put $x = 0$ in the co-ordinates of R, since $x = 0$ for yz-plane.

Hence, $\frac{3\lambda-2}{\lambda+1} = 0 \Rightarrow 3\lambda = 2$ or $\lambda = \frac{2}{3}$. Required ratio is 2 : 3.

Example 13 : **Find the co-ordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 internally.**

SOLUTION :

(i) Let $P(x, y, z)$ be the point which divides line segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ internally in the ratio 2 : 3.

Therefore, $x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5}$, $y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5}$, $z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$

Example 14 : **Using section formula, prove that the three points (– 4, 6, 10), (2, 4, 6) and (14, 0, –2) are collinear.**

SOLUTION : Let $A(-4, 6, 10)$, $B(2, 4, 6)$ and $C(14, 0, -2)$ be the given points. Let the point P divides AB in the ratio $k : 1$. Then co-ordinates of the point P are $\left(\frac{2k-4}{k+1}, \frac{4k+6}{k+1}, \frac{6k+10}{k+1}\right)$

Let us examine whether for some value of k, the point P coincides with point C.

On putting $\frac{2k-4}{k+1} = 14$, we get $k = -\frac{3}{2}$

When $k = -\frac{3}{2}$ then $\frac{4k+6}{k+1} = \frac{4(-3/2)+6}{-\frac{3}{2}+1} = 0$

and $\frac{6k+10}{k+1} = \frac{6(-3/2)+10}{-\frac{3}{2}+1} = -2$

Therefore, $C(14, 0, -2)$ is a point which divides AB externally in the ratio 3 : 2 and is same as P. Hence A, B, C are collinear.

Example 15 : **Prove that the three points *P*, *Q*, *R* whose coordinates are respectively (2, 5, – 4), (1, 4, – 3), (4, 7, – 6) are collinear and find the ratio in which the point *Q* divides *PR*.**

SOLUTION : We can prove that collinearity of the points *P*, *Q*, *R* by showing that $PQ + PR = QR$, so that the point *P* lies on (within) the segment of the line *QR*. Alternatively, we may proceed as follows.

Supposing that the points *P*, *Q*, *R* are collinear, let the point *Q* divide the line segment *PR* in the ratio $m_1 : m_2$.

Then the coordinates of *Q* are $\left(\frac{4m_1+2m_2}{m_1+m_2}, \frac{7m_1+5m_2}{m_1+m_2}, \frac{-6m_1-4m_2}{m_1+m_2}\right)$

We can find the ratio $m_1 : m_2$ by equating any one of these co-ordinates to the given co-ordinates of *Q*. Thus equating the *x*-co-ordinates, we get $4m_1 + 2m_2 = m_1 + m_2$, where $m_1 : m_2 = -1 : 3$

It is necessary to verify that the same result is obtained by equating the other two co-ordinates. Then only it will follow that the assumption of the collinearity of *P*, *Q*, *R* is correct.

Example 16 : **Show that the points *A* (1, 2, 3), *B* (–1, –2, –1), *C* (2, 3, 2) and *D* (4, 7, 6) are the vertices of a parallelogram *ABCD*, but it is not a rectangle.**

SOLUTION : To show *ABCD* is a parallelogram we need to show opposite side are equal

$$AB = \sqrt{(-1-1)^2+(-2-2)^2+(-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2+(3+2)^2+(2+1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2+(7-3)^2+(6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2+(2-7)^2+(3-6)^2} = \sqrt{9+25+9} = \sqrt{43}$$

Since, $AB = CD$ and $BC = AD$, *ABCD* is a parallelogram.

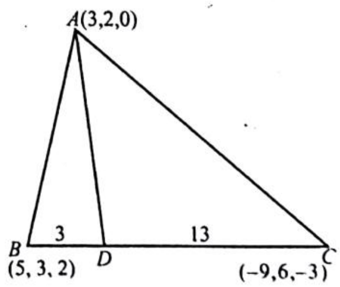
Now, it is required to prove that *ABCD* is not a rectangle. For this, we show that diagonals *AC* and *BD* are unequal. We have

$$AC = \sqrt{(2-1)^2+(3-2)^2+(2-3)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$BD = \sqrt{(4+1)^2+(7+2)^2+(6+1)^2} = \sqrt{25+81+49} = \sqrt{155}$$

Since $AC \neq BD$, *ABCD* is not a rectangle.

Example 17 : ***A* (3, 2, 0), *B* (5, 3, 2), *C* (–9, 6, –3) are three points forming a triangle. *AD*, the bisector of angle *BAC* meets *BC* in *D*. Find the co-ordiantes of *D*.**



SOLUTION : Since *AD* is the bisector of $\angle BAC$.

$$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC} \quad \text{........ (1)}$$

Now, $AB = \sqrt{(5-3)^2+(3-2)^2+(2-0)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

$$AC = \sqrt{(-9-3)^2+(6-2)^2+(-3-0)^2} = \sqrt{144+16+9} = 13$$

$$\therefore \frac{BD}{DC} = \frac{3}{13}$$

Hence, *D* divides *BC* in the ratio 3 : 13

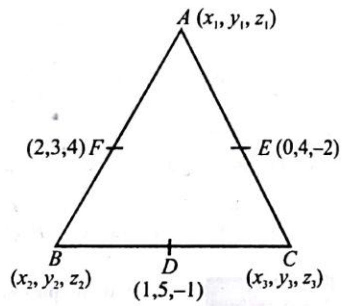
The co-ordinates of *D* are

$$\left(\frac{3(-9)+13(5)}{3+13}, \frac{3(6)+13(3)}{3+13}, \frac{3(-3)+13(2)}{3+13}\right) = \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

Example 18: **The mid-points of the sides of a triangle are (1, 5, –1), (0, 4, – 2) and (2, 3, 4). Find its vertices.**

SOLUTION : Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of the triangle.

Let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the mid-points of the side BC, CA and AB respectively. Then, we have D is the mid-point of BC



$$\Rightarrow \frac{x_2+x_3}{2}=1, \frac{y_2+y_3}{2}=5, \frac{z_2+z_3}{2}=-1$$

$$\Rightarrow x_2+x_3=2,\ y_2+y_3=10,\ z_2+z_3=-2 \quad \text{......... (1)}$$

E is the mid-point of CA

$$\Rightarrow \frac{x_1+x_3}{2}=0, \frac{y_1+y_3}{2}=4, \frac{z_1+z_3}{2}=-2$$

$$\Rightarrow x_1+x_3=0,\ y_1+y_3=8,\ z_1+z_3=-4 \quad \text{......... (2)}$$

F is the mid-point of AB

$$\Rightarrow \frac{x_1+x_2}{2}=2, \frac{y_1+y_2}{2}=3, \frac{z_1+z_2}{2}=4$$

$$\Rightarrow x_1+x_2=4,\ y_1+y_2=6,\ z_1+z_2=8 \quad \text{......... (3)}$$

Adding first three equations (1), (2) and (3), we get

$$2(x_1+x_2+x_3)=2+0+4 \Rightarrow x_1+x_2+x_3=3 \quad \text{......... (4)}$$

Solving first three equations (1), (2) and (3), with (4), we get

$$x_1=1, x_2=3, x_3=-1$$

Adding second three equations (1), (2) and (3), we get

$$2(y_1+y_2+y_3)=10+8+6 \Rightarrow y_1+y_2+y_3=12 \quad \text{......... (5)}$$

Solving second three equations (1), (2) and (3), with (5), we get

$$y_1=2, y_2=4, y_3=6$$

Adding last three equations (1), (2) and (3), we get

$$2(z_1+z_2+z_3)=-2-4+8 \Rightarrow z_1+z_2+z_3=1 \quad \text{......... (6)}$$

Solving last three equations (1), (2) and (3), with (6), we get

$$z_1=3, z_2=5, z_3=-7$$

Hence, the vertices of the triangle are

$A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$.

EXERCISE 1

FIB Fill in the Blanks

DIRECTIONS: *Complete the following statements with an appropriate word/term to be filled in the blank space(s).*

1. The x-axis and y-axis taken together determine a plane known as ....
2. The co-ordinates of points in the xy-plane are of the form ....
3. Co-ordinate planes divide the space into .... octants.
4. Any point on x-axis is of the form ....
5. Any point on y-axis is of the form ....
6. Any point on z-axis is of the form ....
7. The co-ordinates of the mid-point of the line segement joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are (.............., ..., ...)
8. The co-ordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are (.............., ..., ...)
9. Points (3, 2), (–2, –3) and (2, 3) form a .... triangle.
10. If $x - y = 2$ then point (x, y) is equidistant from (7, 1) and (.........)
11. Area of a rhombus if its vertices are (3, 0), (4, 5), (– 1, 4) and (– 2, – 1) taken in order is ....
12. Area of a triangle formed by the points $A(5, 2)$, $B(4, 7)$ and $C(7, -4)$ is ....
13. Point on the x-axis which is equidistant from (2, –5) and (–2, 9) is ....

T/F True / False

DIRECTIONS: *Read the following statements and write your answer as true or false.*

1. The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2+x_1)^2+(y_2+y_1)^2}$.
2. The coordinates of the point $P(x, y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are $\left(\frac{m_1x_2 - m_2x_1}{m_1+m_2}, \frac{m_1y_2 - m_2y_1}{m_1+m_2}\right)$
3. The mid-point of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$.
4. Points (1, 7), (4, 2), (–1, –1) and (– 4, 4) are the vertices of a square.
5. Area of the triangle formed by the points $P(-1.5, 3)$, $Q(6, -2)$ and $R(-3, 4)$ is 0.
6. The ratio in which the point (3, 5) divides the join of (1, 3) and (4, 6) is 2 : 1
7. The distance of the point (5, 3) from the x-axis is 5 units
8. The y-axis and z-axis, together determine a plane known as yz-plane.
9. The point (4, 5, – 6) lies in the 6th octant.
10. The x-axis is the intersection of two planes xy-plane and xz-plane.
11. The equation of the plane $z = 6$ represent a plane parallel to the xy-plane, having a z-intercept of 6 units.
12. The equation of the plane $x = 0$ represent the yz-plane.
13. $x = x_0$ represent a plane parallel to the yz-plane.

MTC Match the Columns

DIRECTIONS: *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D, ...) in Column I have to be matched with statements (p, q, r, s,.....) in column II.*

1.

Column I	Column II
(A) If the centroid of the triangle is origin and two of its vertices are (3, –5, 7) and (–1, 7, – 6) then the third vertex is	(p) Parallelogram
(B) If the mid-points of the sides of triangle are (1, 2, – 3), (3, 0, 1) and (–1, 1, – 4) then the centroid is	(q) (–2, –2, –1)
(C) The points (3, –1, –1), (5, – 4, 0), (2, 3, –2) and (0, 6, – 3) are the vertices of a	(r) Isosceles right-angled triangle
(D) Point A(1, –1, 3), B(2, – 4, 5) and C(5, –13, 11) are	(s) (1, 1, –2)
(E) Points A(2, 4, 3), B(4, 1, 9) and C(10, –1, 6) are the vertices of	(t) Collinear

2.

Column-I	Column-II
(A) In xy-plane	(p) Ist octant
(B) Point (2, 3, 4) lies in the	(q) yz-plane
(C) Locus of the points having x-co-ordinate 0 is	(r) z-coordinate is zero
(D) A line is parallel to x-axis if and only if	(s) z-axis
(E) If $x = 0$, $y = 0$ taken together will represent the	(t) plane parallel to xy-plane

(F)	$z = c$ represent the plane	(u)	all the points on the line have equal y and z-co-ordinates
(G)	Planes $x = a, y = b$ represent the line	(v)	from the point on the respective
(H)	Co-ordinates of a point are the distances from the origin to the feet of perpendiculars	(w)	parallel to z-axis.
(I)	A ball is the solid region in the space enclosed by a	(x)	disc
(J)	Region in the plane enclosed by a circle is known as a	(y)	sphere

3. Column II gives the coordinates of the point P that divides the line segment joining the points given in column I, match them correctly.

Column I	Column II
(A) $A(-1, 3)$ and $B(-5, 6)$ internally in the ratio 1 : 2	(p) (7, 3)
(B) $A(-2, 1)$ and $B(1, 4)$ internally in the ratio 2 : 1	(q) (0, 3)
(C) $A(-1, 7)$ and $B(4, -3)$ internally in the ratio 2 : 3	(r) (1, 3)
(D) $A(4, -3)$ and $B(8, 5)$ internally in the ratio 3 : 1	(s) (1, 0)

VSAQ Very Short Answer Questions

DIRECTIONS: *Give answer in one word or one sentence.*

1. A point is on the x-axis. What are its y-co-ordinate and z-co-ordinates?
2. A point is in the xz-plane. What can you say about its y-co-ordinate?
3. Name the octants in which the following points lie: (1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (–3, –1, 6), (2, – 4, –7).
4. Find the distance between the points (–1, 3, – 4) and (1, –3, 4).
5. Find the co-ordinate of the point P which is five-sixth of the way from $A(-2, 0, 6)$ to $B(10, -6, -12)$.
6. Let L, M, N be the feet of the perpendiculars drawn from a point $P(3, 4, 5)$ on the x, y and z-axes respectively. Find the co-ordinates of L, M and N.
7. L, is the foot of the perpendicular drawn from a point $P(3, 4, 5)$ on the xz-plane. What are the co-ordinates of point L?
8. What is the locus of a point for which $y = 0, z = 0$?
9. If the distance between the points $(a, 0, 1)$ and $(0, 1, 2)$ is $\sqrt{2}$ then find the value of 'a'.
10. If distance between the point $(x, 2)$ and $(3, 4)$ is 2, then find the value of x.
11. Find the ratio in which the line $x + y = 4$ divides the line joining the points (–1, 1) and (5, 7).

SAQ Short Answer Questions

DIRECTIONS: *Give answer in 2-3 sentences.*

1. Using distance formula show that the points $P(2, 4, 6)$, $Q(-2, -2, -2)$ and $R(6, 10, 14)$ are collinear.
2. Show that the points (5, –1, 1), (7, – 4, 7), (1, –6, 10) and (–1, –3, 4) are the vertices of a rhombus.
3. Find the point on y-axis which is at a distance $\sqrt{10}$ from the point (1, 2, 3).
4. Show that the points $A(5, 6)$, $B(1, 5)$, $C(2, 1)$ and $D(6, 2)$ are the vertices of a square.
5. Determine the ratio in which the point P (m, 6) divides the join of A (– 4, 3) and B (2, 8). Also find the value of m.
6. The co-ordinates of the mid-point of a line segment are (2, 3). If co-ordinates of one of the end points of the line segment are (6, 5), find the co-ordinates of the other end point.
7. The vertices of ΔPQR are $P(2, 1)$, $Q(-2, 3)$ and $R(4, 5)$. Find equation of the median through the vertex R.
8. Find the median to the side BC of the triangle whose vertices are $A(-2, 1)$, $B(2, 3)$ and $C(4, 5)$.

LAQ Long Answer Questions

DIRECTIONS: *Give answer in four to five sentences.*

1. Show that the points $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$ are the vertices of a parallelogram $ABCD$, but it is not a rectangle.
2. Find the co-ordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).
3. Find the co-ordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 (i) internally, and (ii) externally.
4. The vertices of a triangle ABC are $A(4, 6)$, $B(1, 5)$ and $C(7,2)$. A line is drawn to intersect sides AB and AC at and E respectively such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of the ΔADE and compare it with the area of ΔABC.
5. Prove that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.
6. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. Find the third vertex.
7. Find the co-ordinates of the points of trisection of the line segment joining (4, – 1) and (–2, –3).

EXERCISE 2

MCQ Multiple Choice Questions

DIRECTIONS: *This section contains multiple choice questions. Each questions has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

1. For every point $P(x, y, z)$ on the xy-plane,
   (a) $x = 0$ (b) $y = 0$
   (c) $z = 0$ (d) none of these
2. For every point $P(x, y, z)$ on the x-axis (except the origin),
   (a) $x = 0, y = 0, z \neq 0$ (b) $x = 0, z = 0, y \neq 0$
   (c) $y = 0, z = 0, x \neq 0$ (d) none of these
3. The distance of the point $P(a, b, c)$ from the x-axis is
   (a) $\sqrt{b^2+c^2}$ (b) $\sqrt{a^2+c^2}$
   (c) $\sqrt{a^2+b^2}$ (d) none of these
4. Are the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1)
   (a) collinear
   (b) vertices of an isosceles triangle
   (c) vertices of a right angled triangle
   (d) none of these
5. The co-ordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 internally is
   (a) $\left(\frac{5}{9}, \frac{5}{2}, \frac{-5}{1}\right)$ (b) $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$
   (c) $\left(\frac{-1}{5}, \frac{2}{5}, \frac{9}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{9}{5}, \frac{-1}{5}\right)$
6. Point (–3, 1, 2) lies in
   (a) Octant I (b) Octant II
   (c) Octant III (d) Octant IV
7. C is the mid-point of PQ, if P is $(4, x)$, C is $(y, -1)$ and Q is $(-2, 4)$, then x and y respectively are –
   (a) – 6 and 1 (b) – 6 and 2
   (c) 6 and – 1 (d) 6 and – 2
8. The ratio in which the point $(2, y)$ divides the join of (– 4, 3) and (6, 3) and hence the value of y is
   (a) $2 : 3, y = 3$ (b) $3 : 2, y = 4$
   (c) $3 : 2, y = 3$ (d) $3 : 2, y = 2$
9. The point which divides the line joining the points A (1, 2) and B(–1, 1) internally in the ratio 1 : 2 is
   (a) $\left(\frac{-1}{3}, \frac{5}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{5}{3}\right)$
   (c) (–1, 5) (d) (1, 5)
10. The area of the triangle formed by the line $5x - 3y + 15 = 0$ with co-ordinate axes is
   (a) 15 cm^2 (b) 5 cm^2
   (c) 8 cm^2 (d) $\frac{15}{2}$ cm^2
11. The points A (– 4, – 1), B (–2, – 4), C (4, 0) and D (2, 3) are the vertices of a –
   (a) Parallelogram (b) Rectangle
   (c) Rhombus (d) Square

MTOC More than One Correct

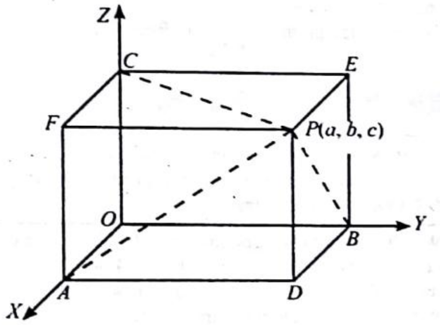
DIRECTIONS: *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which ONE OR MORE may be correct.*

1. Which of the following points is not 10 units from the origin?
   (a) (– 6, 8) (b) (– 4, – 6)
   (c) (– 6, – 8) (d) (6, 4)
2. Which of the following is/are incorrect?
   (a) Image of (–2, 3, 4) in the yz-plane is (2, 3, 4).
   (b) Image of (–2, 3, 4) in the yz-plane is (2, –3, 4).
   (c) Image of (–5, 4, –3) in the xz-plane is (5, 4, 3).
   (d) Image of (–5, 4, –3) in the xz-plane is (–5, –4, –3).
3. Which of the following is/are correct?
   (a) Distance between the points P(–2, 4, 1) and Q(1, 2, –5) is 7 units.
   (b) Image of (– 4, 0, 0) in the xy-plane is (– 4, 0, 0).
   (c) Distance between the points P(–2, 4, 1) and Q(1, 2, –5) is –7 units.
   (d) Image of (– 4, 0, 0) in the xy-plane is (4, 0, 0)
4. Which of the following is/are incorrect?
   (a) In 2-dimensional, x-co-ordinate is –ve and y-co-ordinate is –ve in second quadrant.
   (b) In 3-dimensional, if a point P lies in xy-plane, then z-co-ordinate of P is negative.
   (c) In 3-dimensional, if a point lies on the x-axis, then its y and z-co-ordinates are both zero.
   (d) In 2-dimensional, both x and y-co-ordinates of any point are +ve in third quadrant.
5. Which of the following is/are correct?
   (a) In two dimensional geometry that two mutually perpendicular lines divide the plane containing them into four parts which are known as quadrants and the lines are known as the co-ordinate axes.
   (b) In three dimensional geometry that three mutually perpendicular lines divide the space into eight parts known as octants and the lines are known as the co-ordinate axes.
   (c) In two dimensional goemetry, the abscissa and ordinate of a given point are the distances of the point from y-axis and x-axis respectively.
   (d) In three dimensional geometry, the co-ordinates of a point are also the distances from the origin of the feet of the perpendiculars from the point on the respective co-ordinate axes.

PBQ Passage Based Questions

DIRECTIONS: *Study the given paragraph(s) and answer the following questions.*

In the given figure, if the co-ordinates of point P are (a, b, c) then



(i) The co-ordinate of point A is
(a) $(0, a, 0)$ (b) $(0, 0, a)$
(c) $(a, 0, 0)$ (d) $(-a, 0, 0)$

(ii) The co-ordinate of point D is
(a) $(a, 0, b)$ (b) $(b, 0, a)$
(c) $(b, a, 0)$ (d) $(a, b, 0)$

(iii) The co-ordinate of point E is
(a) $(0, b, c)$ (b) $(b, 0, c)$
(c) $(b, c, 0)$ (d) $(0, c, b)$

A&R Assertion & Reason

DIRECTIONS: *Each of these questions contains an Assertion followed by Reason. Read them carefully and answer the question on the basis of following options. You have to select the one that best describes the two statements.*

(a) If both **Assertion** and **Reason** are **correct** and Reason is the **correct explanation** of Assertion.
(b) If both **Assertion** and **Reason** are correct, but Reason is **not the correct explanation** of Assertion.
(c) If **Assertion** is **correct** but **Reason** is **incorrect**.
(d) If **Assertion** is **incorrect** but **Reason** is **correct**.

1. **Assertion:** The co-ordinates of the point which divides the join of $P(2, -1, 4)$ and $Q(4, 3, 2)$ in the ratio $2 : 3$ externally is $(-2, -9, 8)$.
Reason: If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced dividing it externally in the ratio $m_1 : m_2 (m_1 \neq m_2)$. Then the co-ordinates of R are
$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$$

2. **Assertion:** The distance of a point $P(x, y, z)$ from the origin $O(0, 0, 0)$ is given by
$$OP = \sqrt{x^2 + y^2 + z^2}$$
Reason: The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3. **Assertion:** If $P\left(\frac{a}{3}, 4\right)$ is the mid-point of the line segment joining the points $Q(-6, 5)$ and $R(-2, 3)$, then the value of 'a' is -12.
Reason: The co-ordinates of the mid-point of the line-segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

4. **Assertion:** $P(-2, 3, 5)$, $Q(1, 2, 3)$ and $R(7, 0, -1)$ are collinear points.
Reason: If P, Q and R are three points and satisfies the condition $PQ + QR = PR$ then P, Q, R are said to be collinear points.

5. **Assertion:** The co-ordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are
$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$
Reason: If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle then co-ordinates of centroid of the triangle is
$$\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}\right).$$

SOLUTIONS

EXERCISE – I

FILL IN THE BLANKS

1. xy-Plane 2. $(x, y, 0)$ 3. Eight
4. $(x, 0, 0)$ 5. $(0, y, 0)$ 6. $(0, 0, z)$
7. $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
8. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3},\right)$
9. right angle 10. (3, 5)
11. 24 sq. units 12. 2 sq. units 13. (–7, 0)

TRUE/FALSE

1. False 2. False 3. True 4. True
5. True 6. True 7. False 8. True
9. False 10. True 11. True 12. True
13. True

MATCH THE COLUMN

1. (A) →q; (B) →s; (C) →p; (D) →t; (E) →r

(A) Let A(3, –5, 7), B(–1, 7, –6), C(x, y, z) be the vertices of a ΔABC with centroid (0, 0, 0)

$$\therefore (0,0,0) = \left(\frac{3-1+x}{3}, \frac{-5+7+y}{3}, \frac{7-6+z}{3}\right).$$

$$\Rightarrow \frac{x+2}{3} = 0, \frac{y+2}{3} = 0, \frac{z+1}{3} = 0.$$

Hence $x = -2, y = -2$, and $z = -1$.

(B) Let ABC be the given Δ and DEF be the mid-points of the sides BC, CA, AB, respectively. Centroid of the ΔABC = centroid of ΔDEF.

$\therefore$ centroid of ΔDEF is

$$\left(\frac{1+3-1}{3}, \frac{2+0+1}{3}, \frac{-3+1-4}{3}\right) = (1, 1, -2)$$

(C) Mid-point of diagonal AC is

$$\left(\frac{3+2}{2}, \frac{-1+3}{2}, \frac{-1-2}{2}\right) = \left(\frac{5}{2}, 1, \frac{-3}{2}\right)$$

Mid-point of diagonal BD is

$$\left(\frac{5+0}{2}, \frac{-4+6}{2}, \frac{0-3}{2}\right) = \left(\frac{5}{2}, 1, \frac{-3}{2}\right)$$

Diagonals of parallelogram bisect each other.

(D) $|AB| = \sqrt{(2-1)^2+(-4+1)^2+(5-3)^2} = \sqrt{14}$

$|BC| = \sqrt{(5-2)^2+(-13+4)^2+(11-5)^2} = 3\sqrt{14}$

$|AC| = \sqrt{(5-1)^2+(-13+1)^2+(11-3)^2} = 4\sqrt{14}$

Now $|AB|+|BC| = |AC|$. Hence points A, B, C are collinear.

(E) $AB = \sqrt{4+9+36} = 7$

$BC = \sqrt{36+4+9} = 7$

$CA = \sqrt{64+25+9} = 7\sqrt{2}$

Now $AB^2 + BC^2 = AC^2$. Hence ABC is an isosceles right angled triangle.

2. (A) → r; (B) → p; (C) → q ; (D) → u; (E) → s; (F) → t; (G) → w; (H) → v; (I) → y; (J) → x

3. (A) → s; (B) → q; (C) → r; (D) → p

VERY SHORT ANSWER QUESTIONS

1. y and z-co-ordinates are zero.
2. y-co-ordinate is zero.
3. (1, 2, 3) → Octant I
(4, –2, 3) → Octant IV
(4, –2, –5) → Octant VIII
(4, 2, –5) → Octant V
(–4, 2, –5) → Octant VI
(–3, –1, 6) → Octant III
(2, –4, –7) → Octant VII
4. Required distance

$$= \sqrt{(1+1)^2+(-3-3)^2+(4+4)^2}$$

$$= \sqrt{4+36+64}$$

$$= \sqrt{104} = 2\sqrt{26}$$

5. Let $P(x, y, z)$ be the required point, i.e., P divides AB in the ratio 5 : 1. Then

$$P(x, y, z) = \left(\frac{5\times10+1\times-2}{5+1}, \frac{5\times-6+1\times0}{5+1}, \frac{5\times-12+1\times6}{5+1}\right)$$

$= (8, -5, -9)$

6. Since L is the foot of perpendicular from P on the x-axis, its y and z co-ordinates are zero. The coordinates of L is (3, 0, 0). Similarly, the coordinates of M and N are (0, 4, 0) and (0, 0, 5), respectively.

7. Since L is the foot of perpendicular segment drawn from the point $P(3, 4, 5)$ on the xz-plane. Since the y-coordinate of all points in the xz-plane are zero, coordinate of the foot of perpendicular are (3, 0, 5).

8. Locus of the point $y = 0$, $z = 0$ is x-axis since on x-axis both $y = 0$ and $z = 0$.

9. $a = \pm 5$

10. $2 = \sqrt{(x-3)^2 + (2-4)^2} \Rightarrow 2 = \sqrt{(x-3)^2 + 4}$.

Squaring both sides

$4 = (x-3)^2 + 4 \Rightarrow x - 3 = 0 \Rightarrow x = 3$

11. Ratio $= -\left(\frac{-1+1-4}{5+7-4}\right) = \frac{1}{2}$

SHORT ANSWER QUESTIONS

1. Three points are collinear if the sum of any two distances is equal to the third distance.

$$PQ = \sqrt{(-2,-2)^2 + (-2-4)^2 + (-2-6)^2}$$

$$= \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

$$QR = \sqrt{(6+2)^2 + (10+2)^2 + (14+2)^2}$$

$$= \sqrt{64+144+256} = \sqrt{464} = 4\sqrt{29}$$

$$PR = \sqrt{(6-2)^2 + (10-4)^2 + (14-6)^2}$$

$$= \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

Since $QR = PQ + PR$. Therefore, the given points are collinear.

2. Let $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$ be the four points of a quadrilateral. Here

$AB = \sqrt{4+9+36} = 7$, $BC = \sqrt{36+4+9} = 7$

$CD = \sqrt{4+9+36} = 7$, $DA = \sqrt{23+4+9} = 7$

Since, $AB = BC = CD = DA$. Therefore, $ABCD$ is a rhombus.

3. Let the point P be on y-axis. Therefore, it is of the form $P(0, y, 0)$.

The point (1, 2, 3) is at a distance $\sqrt{10}$ from $(0, y, 0)$.

$\therefore \sqrt{(1-0)^2 + (2-y)^2 + (3-0)^2} = \sqrt{10}$

$\Rightarrow y^2 - 4y + 4 = 0 \Rightarrow (y-2)^2 = 0 \Rightarrow y = 2$

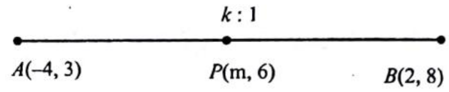
Hence, the required point is (0, 2, 0).

4. $AB = BC = CD = AD$

$\therefore AC^2 = AB^2 + BC^2 \Rightarrow \angle ABC = 90°$

$\therefore ABCD$ is a square.

5. $A(-4, 3)$, $B(2, 8)$ and $P(m, 6)$



Let P divides the join of AB in the ratio of $k : 1$

$\therefore$ y-co-ordinate of $P = \frac{k\times8+1\times3}{k+1}$

$\Rightarrow 6 = \frac{8k+3}{k+1} \Rightarrow 6k + 6 = 8k + 3$

$\Rightarrow 2k = 3 \Rightarrow k = 3/2$

$\therefore$ P divides the join of AB in the ratio of 3 : 2.

x-co-ordiante of $P = \frac{3\times2+2\times(-4)}{3+2}$

$\Rightarrow m = \frac{6-8}{5} \Rightarrow m = -2/5$

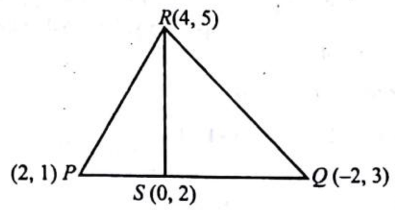
6. (–2, 1) is the co-ordinates of the other end point.

7. The vertices P and Q are (2, 1) and (–2, 3) respectively.

The middle point is $\left(\frac{2-2}{2}, \frac{1+3}{2}\right)$ or (0, 2)

$\therefore$ Equation of the median RS, where R is (4, 5) and S is the point (0, 2), is

$$y - 5 = \frac{2-5}{0-4}(x-4) \text{ or } y - 5 = \frac{-3}{-4}(x-4)$$



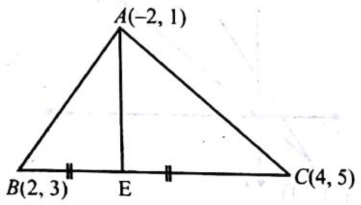
or $4(y-5) = 3(x-4)$ or $4y - 20 = 3x - 12$

i.e. $3x - 4y = -8$

$\therefore$ Equation of median RS is $3x - 4y + 8 = 0$

8. Let E be the mid-point of side BC.

$$\therefore \quad E = \left(\frac{2+4}{2}, \frac{3+5}{2}\right) = (3, 4)$$



Equation of line AE is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

i.e., $y - 1 = \dfrac{4-1}{3-(-2)}(x-1)$

$$y - 1 = \frac{3}{5}(x-1)$$

$$5y - 5 = 3x - 3$$

$\therefore$ The required equation of the median is

$$3x - 5y + 2 = 0$$

LONG ANSWER QUESTIONS

1. To show $ABCD$ is a parallelogram we need to show opposite side are equal.

$$AB = \sqrt{(-1-1)^2 + (-2-2)^2 + (-1-3)^2}$$

$$= \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2 + (3+2)^2 + (2+1)^2}$$

$$= \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2}$$

$$= \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2 + (2-7)^2 + (3-6)^2}$$

$$= \sqrt{9+25+9} = \sqrt{43}$$

Since $AB = CD$ and $BC = AD$, $ABCD$ is a parallelogram.

Now, it is required to prove that $ABCD$ is not a rectangle. For this, we show that diagonals AC and BD are unequal.

We have

$$AC = \sqrt{(2-1)^2 + (3-2)^2 + (2-3)^2}$$

$$= \sqrt{1+1+1} = \sqrt{3}$$

$$BD = \sqrt{(4+1)^2 + (7+2)^2 + (6+1)^2}$$

$$= \sqrt{25+81+49} = \sqrt{155}$$

Since $AC \neq BD$, $ABCD$ is not a rectangle.

2. Let ABC be the triangle. Let the co-ordinates of the vertices A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), respectively. Let D be the mid-point of BC. Hence coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

Let G be the centroid of the triangle. Therefore, it divides the median AD in the ratio 2 :1. Hence, the coordinates of G are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}, \frac{2\left(\frac{z_2+z_3}{2}\right)+z_1}{2+1}\right)$$

or $\quad \left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}, \dfrac{z_1+z_2+z_3}{3}\right)$.

3. (i) Let $P(x, y, z)$ be the point which divides line segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ internally in the ratio 2 : 3. Therefore

$$x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5}, \quad y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5},$$

$$z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\dfrac{9}{5}, \dfrac{2}{5}, \dfrac{-1}{5}\right)$

(ii) Let $P(x, y, z)$ be the point which divides segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ externally in the ratio 2 : 3. Then

$$x = \frac{2(3)+(-3)(1)}{2+(-3)} = -3,$$

$$y = \frac{2(4)+(-3)(-2)}{2+(-3)} = -14,$$

$$z = \frac{2(-5)+(-3)(3)}{2+(-3)} = 19$$

Therefore, the required point is (−3, −14, 19).

4. $\frac{AD}{AB} = \frac{1}{4}$

$\Rightarrow \quad 4AD = AD + BD \Rightarrow 3AD = DB$

$\Rightarrow \quad \frac{AD}{DB} = \frac{1}{3}$

D divides AB in the ratio 1 : 3.

By section formula, the coordinates of D are $\left(\frac{13}{4}, \frac{23}{4}\right)$.

Similarly, E divides AC in the ratio 1 : 3.

The co-ordinates of E are $\left(\frac{19}{4}, 5\right)$.

Area of ΔADE

$= \frac{1}{2}\left|4\left(\frac{23}{4} - 5\right) + \frac{13}{4}(5-6) + \frac{19}{4}\left(6 - \frac{23}{4}\right)\right|$ sq. units

$= \frac{1}{2}\left|3 - \frac{13}{4} + \frac{19}{4}\right|$ sq. units $= \frac{15}{32}$ sq. units

Area of ΔABC

$= \frac{1}{2}|4(5-2) + 1(2-6) + 7(6-5)|$ sq. units

$= \frac{1}{2}|12 - 4 + 7|$ sq. units $= \frac{15}{2}$ sq. units

$\therefore \quad \frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = \frac{15}{32} \times \frac{2}{15} = \frac{1}{16}$

5. Let the points (4, 4), (3, 5) and (–1, –1) be denoted by P, Q and R, respectively.

Now $PQ = \sqrt{(3-4)^2 + (5-4)^2} = \sqrt{2}$

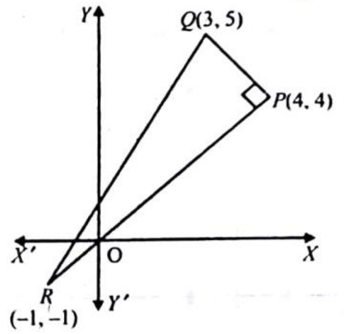
$QR = \sqrt{(-1-3)^2 + (-1-5)^2} = \sqrt{52}$

and $PR = \sqrt{(-1-4)^2 + (-1-4)^2} = \sqrt{50}$

Therefore, $PQ^2 = 2$, $QR^2 = 52$ and $PR^2 = 50$

We observe that the sum of square of two sides, PQ and PR, is equal t the square of the third side QR i.e.,

$QR^2 = PR^2 + PQ^2$



If follows from the converse of the Pythagoras theorem that the triangle PQR is a right triangle and the right angle is at P.

6. Let the third vertex be (x_3, y_3). Area of triangle , whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$= \frac{1}{2}|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$

As $x_1 = 2, y_1 = 1, x_2 = 3, y_2 = -2$

Area of $\Delta = 5$ (given)

$\Rightarrow \quad 5 = \frac{1}{2}|2(-2 - y_3) + 3(y_3 - 1) + x_3(1+2)|$

$\Rightarrow \quad 10 = |3x_3 + y_3 - 7|$

Taking positive sign, $3x_3 + y_3 - 7 = 10$

$\Rightarrow \quad 3x_3 + y_3 = 17$ .... (1)

Taking negative sign $3x_3 + y_3 - 7 = -10$

$\Rightarrow \quad 3x_3 + y_3 = -3$ .... (2)

Given that (x_3, y_3) lies on $y = x + 3$

So, $-x_3 + y_3 = 3$ .... (3)

Solving eq. (1) and (3), $x_3 = \frac{7}{2}$, $y_3 = \frac{13}{2}$

Solving eq. (2) and (3), $x_3 = \frac{-3}{2}$, $y_3 = \frac{3}{2}$.

So, the third vertex are $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(\frac{-3}{2}, \frac{3}{2}\right)$

7. Let P and Q are points of trisection (points dividing in three equal parts) of AB, i.e. $AP = PQ = QB$.

The point P divides AB internally in the ratio 1 : 2.

Therefore, the co-ordinates of P will be given by :

$$x = \frac{m_2x_1 + m_1x_2}{m_1 + m_2} = \frac{2 \times 4 + 1(-2)}{1+2} = 2$$

$$y = \frac{m_2y_1 + m_1y_2}{m_1 + m_2} = \frac{2 \times (-1) + 1(-3)}{1+2} = \frac{-5}{3}$$

Therefore, the co-ordinate of P are $\left(2, -\frac{5}{3}\right)$

For the coordinate of Q, $m_2 = 2$ and $m_1 = 1$,

$$x = \frac{m_2x_1 + m_1x_2}{m_1 + m_2} = \frac{1(4) + 2(-2)}{1+2} = 0$$

$$y = \frac{m_2y_1 + m_1y_2}{m_1 + m_2} = \frac{1(-1) + 2(-3)}{1+2} = -\frac{7}{3}$$

$\therefore$ Point $Q = \left(0, -\frac{7}{3}\right)$

EXERCISE – 2

MULTIPLE CHOICE QUESTIONS

1. (c) On xy-plane, z-co-ordinate is zero.
2. (c) On x-axis, y and z-co-ordinates are zero.
3. (a) Let $(a, 0, 0)$ be a point on x-axis.

   So, required distance

   $$= \sqrt{(a-a^2) + (b-0)^2 + (c-0)^2}$$

   $$= \sqrt{b^2 + c^2}$$
4. (a) Given points $(-2, 3, 5)$, $(1, 2, 3)$ and $(7, 0, -1)$ are collinear because area of triangle formed by these three points is zero.
5. (b) Let $P(x, y, z)$ be the point which divides line-segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ internally in the ratio 2 : 3

   $$\therefore \quad x = \frac{2(3) + 3(1)}{2+3} = \frac{9}{5}$$

   $$y = \frac{2(4) + 3(-2)}{2+3} = \frac{2}{5}$$

   $$z = \frac{2(-5) + 3(3)}{2+3} = \frac{-1}{5}$$

   $\therefore$ Required point $= \left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$
6. (b) $(-3, 1, 2)$ lies in second octant.
7. (a) Since $C(y, -1)$ is the mid-point of $P(4, x)$ and $Q(-2, 4)$.

   We have, $\frac{4-2}{2} = y$ and $\frac{4+x}{2} = -1$

   $\therefore$ $y = 1$ and $x = -6$
8. (c) Let the required ratio be $k : 1$

   Then, $2 = \frac{6k - 4(1)}{k+1}$ or $k = \frac{3}{2}$

   $\therefore$ The required ratio is $\frac{3}{2} : 1$ or $3 : 2$

   Also, $y = \frac{3(3) + 2(3)}{3+2} = 3$
9. (b) 10. (d) 11. (b)

MORE THAN ONE CORRECT

1. (b, d) 2. (b, c) 3. (a, b)
4. (a, b, d) 5. (a, b, c, d)

PASSAGE BASED QUESTIONS

(i) (c) Since the coordinates of P are (a, b, c)

$\therefore$ $OA = a$, $OB = b$ and $OC = c$

Now, A lies on OX such that $OA = a$

So, the coordinates of A are $(a, 0, 0)$

(ii) (d) Since D lies in xy-plane such that $OA = a$ and $AD = OB = b$.

So, the co-ordinates of D are $(a, b, 0)$

(iii) (a) Point E lies in yz-plane such that $OB = b$ and $BE = OC = c$. So, the co-ordinates of E are $(0, b, c)$.

ASSERTION & REASON

1. (a) **Assertion**

$$x = \frac{2\times 4 - 3\times 2}{2-3} \Rightarrow x = -2$$

$$y = \frac{2\times 3 - 3\times -1}{2-3} \Rightarrow y = -9$$

$$z = \frac{2\times 2 - 3\times 4}{2-3} \Rightarrow z = 8$$

2. (a) Both assertion and reason are correct and reason is correct explanation for assertion.

3. (a) **Assertion:**

Since, P is mid-point

$$\therefore \quad \frac{-6+(-2)}{2} = \frac{a}{3} \Rightarrow \frac{-8}{2} = \frac{a}{3} \Rightarrow a = -12$$

Both assertion and reason are correct and reason is correct explanation for assertion.

4. (a) **Assertion:**

$$PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2}$$

$$= \sqrt{9+1+4} = \sqrt{14}$$

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$$

$$= \sqrt{36+4+16} = 2\sqrt{14} \text{ and}$$

$$PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$$

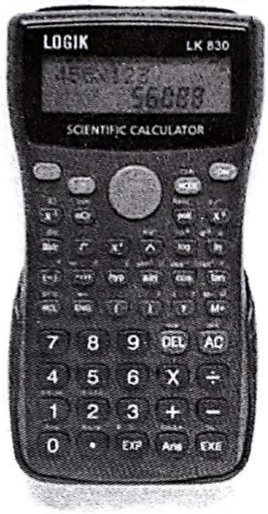
$$= \sqrt{81+9+36} = 3\sqrt{14}$$

Since, $PQ + QR = PR$

therefore P, Q and R are collinear.

5. (c) Assertion is correct.

Reason is incorrect.



Annexure
Logharithm and Antilogarithm

INTRODUCTION

Basic knowledge of logarithm and antilogarithm is very important not only for the point of view of mathematics but for the point of view of physics and chemistry also. It helps to find any root of positive real number and to find the value of a complex arithmetic expressions in mathematics. In physics and chemistry, logarithm is used to define some terms like pH value of a solution, to derive the various formulas and solving many numerical problems.

LOGARITHM

Let us consider

(i) $10 = 10^1$, $100 = 10^2$, $1000 = 10^3$ etc.

This can be also written as

$\log_{10} 10 = 1$, $\log_{10} 100 = 2$, $\log_{10} 1000 = 3$ etc.

(ii) $8 = 2^3$, $16 = 2^4$, $32 = 2^5$ etc.

This can be also written as

$\log_2 8 = 3$, $\log_2 16 = 4$, $\log_2 32 = 5$ etc.

(iii) $49 = 7^2$, $343 = 7^3$ etc.

This can be also written as

$\log_7 49 = 2$, $\log_7 343 = 3$ etc.

(iv) $1/8 = (2)^{-3}$, $1/16 = (2)^{-4}$ etc.

This can be also written as

$\log_2 (1/8) = -3$, $\log_2 (1/16) = -4$ etc.

(v) $0.01 = (10)^{-2}$, $0.001 = (10)^{-3}$ etc.

This can be also written as

$\log_{10} (0.01) = -2$, $\log_{10} 0.001 = -3$ etc.

Here logarithm is written as log in short.

$\log_a x$ is read as logarith of x on base a.

Here x is always a positive number but the base 'a' is a positive number not equal to 1.

DEFINITION OF LOGARITHM

Logarithm of any positive number b ($= a^x$) to a given base 'a' (a positive number not equal to 1) is the index (or power) 'x' of the base which is equal to that number b.

Thus if $b = a^x$, where 'b' is a +ve number and 'a' is a +ve number but not equal to 1, then $\log_a b = x$

Clearly, $\log_a a = 1$

Also, $16 = 2^4$ or 4^2

$\therefore$ $\log_2 16 = 4$ and $\log_4 16 = 2$

$\therefore$ $\log_2 16 \neq \log_4 16$

Hence value of logarithm of a positive quantity depends upon the base of the log.

Base of logarithm is generally taken as '10' or 'e'. 'e' is called an exponential number and is approximately equal to 2.71. In this chapter, if base of log is not mentioned, then base will be taken as 10.

Note : (i) If base of log is not mentioned, then up to secondary classes it is assumed as '10'.

For example, $\log x = \log_{10} x$, $\log 10 = \log_{10} 10\ (= 1)$

log on base 10 is called common log.

But in higher classes, it is assumed as 'e'. For example, $\log x = \log_e x$, $\log e = \log_e e\ (= 1)$

Generally log on base e is written as ℓn, called lognatural. For example, $\log_e x = \ell n\, x$

(ii) $\ell n\, x = 2.303 \log_{10} x$

STANDARD FORM OF A POSITIVE NUMBER

Standard form of a positive number is $m \times 10^n$, where $1 \leq m < 10$ and n is an integer.

S. No.	Number	Number in Standard form upto three decimal place	m	n	Number formed from first two digits of m, without considering the decimal point	Third digit of m, without considering the decimal point	Fourth digit of m, without considering the decimal point
1	243.6	2.436×10^2	2.436	2	24	3	6
2	2436	2.436×10^3	2.436	3	24	3	6
3	6.034	6.034×10^0	6.034	0	60	3	4
4	60.34	6.034×10^1	6.034	1	60	3	4
5	0.6034	$\frac{6.034}{10} = 6.034\times10^{-1}$	6.034	−1	60	3	4
6	0.006034	$\frac{6.034}{1000} = 6.034\times10^{-3}$	6.034	−3	60	3	4
7	56	5.600×10^1	5.600	1	56	0	0
8	0.2	$\frac{2.000}{10} = 2.000\times10^{-1}$	2.000	−1	20	0	0
9	0.6	6.000×10^{-1}	6.000	−1	60	0	0
10	0.00081	$\frac{8.100}{10000} = 8.100\times10^{-4}$	8.100	−4	81	0	0

Note : To find the logarithm of any number using logarithm table, number of digits after decimal point in m should be 3. But if third digit after decimal is 0, then do not consider this 0. And find the logarithm of the number up to two decimal place only.

A PART OF THE LOGARITHM TABLE ON BASE 10

		First 0 to 9										Second 1 to 9								
1st row →		0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
2nd row →	11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
							0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
3rd row →	12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
							0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
4th row →	13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
							1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
5th row →	59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
6th row →	60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
7th row →	61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6

Table (a)

See the above given logarithm table on base 10. The 2nd, 3rd and 4th rows, columns heading first 0 to 9 are divided into two half filled subrows but the columns heading second 1 to 9 are divided into two completely filled half subrows. Remaining 5th, 6th and 7th rows are not divided into subrows.

Complete logarithm table is given at the end of this chapter.

TO FIND THE LOGARITHM OF A NUMBER ON BASE 10 USING LOGARITHM TABLE

Let us consider a number 124.6 of which we want to find the logarithm.

(i) Write the number 124.6 in standard form $m \times 10^n$ as $124.6 = 1.246 \times 10^2$, where $m = 1.246$ and $n = 2$.

(ii) See the logarithm table (a). In the left most column, see the number 12 formed from first two digits of m (= 1.246), without considering its decimal point. This number lies in 3rd row of the logarithm table from the top.

		0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3rd row →	12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
							0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31

Table (b)

(iii) In the first 0 to 9 of first row of the logarithm table (a), see the third digit 4 of m (= 1.246), without considering the decimal point. This third digit lies in 6th column from left.

First 0 to 9
6th column

Second 1 to 9

		0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3rd row →	12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
							0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31

Table (c)

(iv) Now see the number 0934 in the 3rd row and 6th column from left of the logarithm table (a).

First 0 to 9
6th column

Second 1 to 9

		0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3rd row →	12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
							0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31

Table (d)

Note : Do not ignore '0' in 0934, you will see its importance in step - (viii).

(v) In the second 1 to 9 of first row of logarithm table (a), see the fourth digit of '6' of m (= 1.246), without considering the decimal point. This fourth digit '6' lies in 17th column of the logarithm table (a) from the left.

First 0 to 9
6th column

Second 1 to 9
7th column

		0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3rd row →	12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
							0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31

Table (e)

(vi) Now see the number in 3rd row and 17th column of the logarithm table (a), there are two numbers 21 and 20.

First 0 to 9 6th column

Second 1 to 9 7th column

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3rd row → 12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31

Table (f)

But we take one of the two numbers 21 and 20, which lies in the sub row of 0934 obtained in step (iv). We clearly see that the number 21 lies in the sub row of 0934. So, we take 21.

(vii) Now we find the sum of 0934 and 21 as

$$\begin{array}{r} 0934 \\ 21 \\ \hline 0955 \end{array}$$

(viii) Put decimal point in the beginning of the number 0955 as '.0955' and number 2 (value of n) before the decimal point of .0955 as 2.0955.

Note : If the value of n is negative, then negative sign is put over the digit before the decimal point. For example, if the value of n in the above example is -2, then we write $\bar{2}.0955$. $\bar{2}.0955$ means $-2 + .0955$

The number 2.0955 is the logarithm of 124.6 on base 10. Hence, $\log_{10}124.6 = 2.0955$. (or simply $\log 124.6 = 2.0955$)

Note : In the above number 2.0955 has two part, one is integral part, which is 2 and other is non-integral part, which is .0955. The integral part 2 is called characteristic and non-integral part '.0955' is called mantissa. Mantissa is always non-negative.

Illustration1 : Find the value of $\log_{10}0.002001$.

SOLUTION : $\log_{10}0.002001 = \log_{10} 2.001 \times 10^{-3} = \bar{3}.3012$

Illustration 2 : Find the value of log 345

SOLUTION : $\log 345 = \log_{10}345 = \log 3.450 \times 10^2$

$= 2.5378$

LAWS OF LOGARITHM

(i) $\log_b(m \times n) = \log_b m + \log_b n$

For example : $\log (4 \times 5) = \log 4 + \log 5$

(ii) $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$

For example : $\log \frac{8}{15} = \log 8 - \log 15$

(iii) $\log_b(m)^n = n \log_b m$

For example : $\log (5)^4 = 4 \log 5$

(iv) $\log_b a = \dfrac{\log_c a}{\log_c b}$ **[Change of base rule]**

If $a = c$, then $\log_b a = \log_b a = \dfrac{\log_a a}{\log_a b} = \dfrac{1}{\log_a b}$

For example :

(a) $\log_4 64 = \dfrac{\log_2 64}{\log_2 4}$

(b) $\log_4 64 = \dfrac{1}{\log_{64} 4}$

(v) $\log_b a \log_c b = \log_c a$ **[Chain Rule]**

For example : $\log_4 64 \times \log_2 4 = \log_2 64$

ANTILOGARITHM

Antilogarithm is simply called antilog.

Consider, $32 = 2^5$

This can be said that antilog of 5 on base 2 is 32. In other words, antilog of 5 on base 2 is 2^5. Hence antilog of m on base b is b^m.

Since $32 = 2^5$ means log of 32 on base 2 is 5 or antilog of 5 on base 2 is 32.

Hence log and antilog are reverse to each other.

$\therefore$ Antilog $(\log x) = x$

or $\log(\text{antilog } x) = x$

TO FIND THE ANTILOG OF A NUMBER USING ANTILOG TABLE

Let us consider a number '2.4153' of which we want to find the antilogarithm. In '2.4153', there are two parts one is integral and other is non-integral. Here the integral part is '2' and non-integral part is '.4153'. The integral part '2' is called 'characteristic' and non-integral part '.4153' is called 'mantissa'. To find the antilog of a number using antilog table, mantissa should be always non-negative.

If mantissa of a number is negative, than to find its antilog, first we make the mantissa positive.

For example, in the case of a number – 2.5021 (= – 2 – .5021), mantissa is – .5021, which is negative. The mantissa – .5021 will be made positive by adding 1 and subtracting 1 as

$$\begin{aligned} -2.5021 &= -2 - .5021 + 1 - 1 \\ &= (-2 - 1) + (1 - .5021) \\ &= -3 + .4979 \\ &= \bar{3}.4979 \end{aligned}$$

Let us find the antilog of 2.4153

(i) Consider its mantissa .4153.

(ii) Now read the number 2600 which lies in the row in which .41 (first two digits with decimal point of .4153) lies in the left most column and the column in which 5 (third digit of .4153) lies in the first 0 to 9 of first row.

(iii) Now read the number 1 which lies in the row in which 41 lies in the left most column and the column in which 3 (fourth digit of .4153) lies in the second 1 to 9 of first row.

(iv) Now add the number 2600 and 1 as

$$\begin{array}{r} 2600 \\ \underline{\quad 1} \\ 2601 \end{array}$$

(v) Add 1 to the characteristic 2

$2 + 1 = 3$

Note : We always add 1 to the characteristic.

(vi) Here the sum is positive and equal to 3. In this case, we put decimal in 2601, obtained in step (iv) after three (obtained in step (v)) digits from left of it as 260.1. The number 260.1 is the antilog of 2.4153.

Hence antilog (2.4153) = 260.1

If characteristic is negative, say – 4, then after adding 1 to the characteristic, we get – 3, a negative integer, then we put three zeros (000) before 2601 and then put decimal before these three zeros as '.0002601'.

Hence, antilog $(\bar{4}.4153)$ = .0002601

If characteristic is – 1, then after adding 1 to the characteristic, we get zero (0), then we put decimal before '2601' as '.2601'.

Hence, antilog $(\bar{1}.4153)$ = .2601

Illustration 3 : Find the antilog of 0.341

SOLUTION : Antilog (0.341) = 2.193

Illustration 4 : Find the antilog of 6.2134

SOLUTION : Antilog (6.2134) = 163500.00 = 1635000

Illustration 5 : Find the antilog of $\bar{5}.412$.

SOLUTION : Antilog $(\bar{5}.412)$ = .00002582 = 0.00002582

Illustration 6 : If log 2 = 0.3010 and log 3 = 0.4771, then find the value of log 72

SOLUTION : $\log 72 = \log (8 \times 9) = \log 8 + \log 9$ [$\because \log m \times n = \log m + \log n$]

$= \log (2)^3 + \log (3)^2$

$= 3 \log 2 + 2 \log 3$

$= 3 \times 0.3010 + 2 \times 0.4771$

$= 0.9030 + 0.9542$

$= 1.8572$

Illustration 7 : Find the value of $\dfrac{2.135\times0.6021}{0.0014\times314}$

SOLUTION : Let $x = \dfrac{2.135 \times 0.6021}{0.0014 \times 314}$

Taking log on both sides,

$$\log x = \log\frac{2.135 \times 0.6021}{0.0014 \times 314}$$

$= \log (2.135 \times 0.6021) - \log (0.0014 \times 314)$ [$\because \log\frac{a}{b} = \log a - \log b$]

$= \log 2.135 + \log 0.6021 - (\log 0.0014 + \log 314)$ [$\because \log m \times n = \log m - \log n$]

$= \log 2.135 + \log 0.6021 - \log 0.0014 - \log 314$

$= \log 2.135 \times 10^0 + \log 6.021 \times 10^{-1} - \log 1.40 \times 10^{-3} - \log 3.14 \times 10^2$

$= 0.3294 + \bar{1}.7797 - \bar{3}.1461 - 2.4969$

$= 0.3294 + (-1 + 0.7797) - (-3 + 0.1461) - 2.4969$

$= 0.329 - 1 + 0.7797 + 3 - 0.1461 - 2.4969$

$= (0.329 + 0.7797 + 3) - (1 + 0.1461 + 2.4969)$

$\log x = 4.1087 - 3.6430 = 0.4657$

Taking antilog on both sides, we get

Antilog ($\log x$) = Antilog (0.4657)

$\Rightarrow \quad x = \text{Antilog } (0.4657)$ $\quad$ [$\because$ Antilog ($\log x$) = x = log (antilog x)]

$\Rightarrow \quad x = 6.682$

$$\Rightarrow \quad \frac{2.135 \times 0.6021}{0.0014 \times 314} = 6.682$$

Illustration 8 : If 5 log 3 + log x = 5 log 2, then find the value of x.

SOLUTION : $5 \log 3 + \log x = 5 \log 2$

$\Rightarrow \quad \log x = 5 \log 2 - 5 \log 3$

$= \log (2)^5 - \log (3)^5$ $\quad$ [$\because \log (m)^n = n \log m$]

$= \log 32 - \log 243$

$= \log \dfrac{32}{243}$ $\quad$ [$\because \log \dfrac{m}{n} = \log m - \log n$]

Illustration 9 : If $\log_y x = \left(\dfrac{\log_a y}{\log_a x}\right)^k$, then find the value of k

SOLUTION : $\log_y x = (\log_x y)^k$

$$= \left(\frac{1}{\log_y x}\right)^k$$

$(\log_y x)^1 = (\log_y x)^{-k}$

$\therefore \quad -k = 1$

$\Rightarrow \quad k = 1$

Illustration 10 : Find the value of $\dfrac{2 - \log_{10}(10)^3}{\log_6 6}$.

SOLUTION : $\dfrac{2 - \log_{10}(10)^3}{\log_6 6} = \dfrac{2 - 3\log_{10} 10}{1}$ $\quad$ [$\because \log (m)^n = n \log m$]

$= \dfrac{2 - 3 \times 1}{1}$ $\quad$ [$\because \log_a a = 1$]

$= -1$

Illustration 11 : If 2 log x + 2 log$y$ = k and xy = 2, then find the value of k.

SOLUTION : $2 \log x + 2 \log y = k$

$\Rightarrow \quad 2 (\log x + \log) = k$

$\Rightarrow \quad 2 \log (xy) = k$ $\quad$ [$\because \log(m \times n) = \log m + \log n$]

$\Rightarrow \quad 2 \times 2 = k$ $\quad$ [$\because xy = 2$]

$\therefore \quad k = 4$

Illustration 12 : Find the value of $\log_y x \times \log_z y \times \log_x z$

SOLUTION : $\log_y x \times \log_z y \times \log_x z = \log_x z$

$= \log_x x$ $\quad$ [Chain Rule]

$= 1$

Illustration 13 : If antilog (0.2156) = 1.643, then find the values of the antilog (1.2156).

SOLUTION : Antilog (1.2156) = 16.43

Illustration 14 : Express 0.000 000 001 = $(.001)^3$ in logarithmic form.

SOLUTION : $0.000\,000\,001 = (.001)^3$

$\Rightarrow \quad \log_{.001} 0.000\,000\,001 = 3$

Illustration 15 : If $\log_{10} 2 = 0.3010$, find the value of $\log_{10} 5$.

SOLUTION : $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right) + \log_{10}10 - \log_{10} 2$ $\quad [\because \log \frac{m}{n} = \log m - \log n]$

$= 1 - \log_{10} 2$ $\quad [\because \log_a a = 1]$

$= 1 - 0.3010$

$= 0.699$

Illustration 16 : If log 2 = 0.3010, log 3 = 0.4771 and log 7 = 0.8451, then find the value of log 210

SOLUTION : $\log 210 = \log (21 \times 10) = \log 3 \times 7 \times 10$

$= \log 3 + \log 7 + \log 10$ $\quad [\because \log(m \times n) = \log m + \log n]$

$= 0.3010 + 0.8451 + 1$ $\quad [\because \log 10 = \log_{10} 10 = 1]$

$= 2.1461$

Illustration 17 : Prove that log 5040 = 4 log 2 + 2 log 3 + log 5 + log 7

SOLUTION : RHS $= 4 \log 2 + 2 \log 3 + \log 5 + \log 7$

$= \log (2)^4 + \log (3)^2 + \log 5 + \log 7$

$= \log 16 + \log 9 + \log 5 + \log 7$

$= \log (16 \times 9 \times 5 \times 7)$

$= \log 5540$

= RHS

Illustration 18 : Using log table, find the value of $\sqrt{(0.17)^5}$.

SOLUTION : Let $x = \sqrt{(0.17)^5} = [(0.17)^5]^{1/2}$

$\Rightarrow \quad x = (0.17)^{5/2}$

$\Rightarrow \quad \log x = \log (0.17)^{5/2}$

$\Rightarrow \quad \log x = \frac{5}{2} \log (0.17)$

$= \frac{5}{2} \log (1.70 \times 10^{-1})$

$= \frac{5}{2} (\bar{1}.2304)$

$= \frac{5}{2}(-1 + .2304)$

$= \frac{5}{2}(-0.7696)$

$= -5 \times (.3848) = -1.9240$

$\log x = -1 - 0.924 = -1 - 0.924 + 1 - 1$
$= (-1 - 1) + (1 - 0.924)$
$= -2 + 0.076$

$\log x = \bar{2}.076$

Taking antilog on the both sides, we get

antilog $(\log x)$ = antilog $(\bar{2}.076)$

$\Rightarrow \quad x = .01191 = 0.01191$

Illustration 19 : If $\log 2 = 0.3010$, then find the number of digits in $\log (2)^{10000}$.

SOLUTION : $\log (2)^{10000} = 10000 \log 2$
$= 10000 \times 0.3010$
$= 3010$

Hence required number of digits = 4

Illustration 20 : Evaluate $3 - \log_{10} 1000$.

SOLUTION : $3 - \log_{10} 1000 = 3 - \log_{10} (10)^3$
$= 3 - 3 \log_{10} 10$
$= 3 - 3 \times 1 = 0$

Illustration 21 : Find the value of $\log_{\sqrt{2}} 16$

SOLUTION : $\log_{\sqrt{2}} 16 = \log_{\sqrt{2}} (2)^4 = \log_{\sqrt{2}} (\sqrt{2})^8$
$= 8 \log_{\sqrt{2}} \sqrt{2}$
$= 8 \times 1 = 8$

Illustration 22 : If $x^2 + y^2 = 25xy$, then prove that $2 \log (x + y) = 3 \log 3 + \log x + \log y$

SOLUTION : LHS $= 2 \log (x + y)$
$= \log (x + y)^2$
$= \log (x^2 + y^2 + 2xy)$
$= \log (25xy + 2xy)$
$= \log (27xy)$
$= \log (3^3 \times x \times y)$
$= \log (3)^3 + \log x + \log y$
$= 3 \log 3 + \log x + \log y$
= RHS

LOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170						5 9 13	17 21 26	30 34 38
						0212	0253	0294	0334	0374	4 8 12	16 20 24	28 32 36
11	0414	0453	0492	0531	0569						4 8 12	16 20 23	27 31 35
						0607	0645	0682	0719	0755	4 7 11	15 18 22	26 29 33
12	0792	0828	0864	0899	0934						3 7 11	14 18 21	25 28 32
						0969	1004	1038	1072	1106	3 7 10	14 17 20	24 27 31
13	1139	1173	1206	1239	1271						3 6 10	13 16 19	23 26 29
						1303	1335	1367	1399	1430	3 7 10	13 16 19	22 25 29
14	1461	1492	1523	1553	1584						3 6 9	12 15 19	22 25 28
						1614	1644	1673	1703	1732	3 6 9	12 14 17	20 23 26
15	1761	1790	1818	1847	1875						3 6 9	11 14 17	20 23 26
						1903	1931	1959	1987	2014	3 6 8	11 14 17	19 22 25
16	2041	2068	2095	2122	2148						3 6 8	11 14 16	19 22 24
						2175	2201	2227	2253	2279	3 5 8	10 13 16	18 21 23
17	2304	2330	2355	2380	2405						3 5 8	10 13 15	18 20 23
						2430	2455	2480	2504	2529	3 5 8	10 12 15	17 20 22
18	2553	2577	2601	2625	2648						2 5 7	9 12 14	17 19 21
						2672	2695	2718	2742	2765	2 4 7	9 11 14	16 18 21
19	2788	2810	2833	2856	2878						2 4 7	9 11 13	16 18 20
						2900	2923	2945	2967	2989	2 4 6	8 11 13	15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3836	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5052	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5515	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5965	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8

LOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7456	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7557	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8587	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

ANTILOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1469	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1667	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5243	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	3	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	9241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20